VOL. 15, NO. 2 WATER RESOURCES RESEARCH APRIL 1979

Storage Mass-Curve Analysis in a Systems-Analytic Perspective

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During the past decade, the systems approach to storage reservoir problems has been heralded as something of a jump from the stone age of mass-curve analysis into the modern era of science. In reality, however, no such jump ever occurred. There were a number of small ones but, contrary to the common belief, many of them were confined to the staircase of mass-curve analysis and not all of them were in the upward direction. This paper attempts to put the mass-curve technique into a proper perspective by clearing out some undesirable semantic underbrush accumulated over the past decades and by showing an intrinsic identity of some mass-curve and systems-analytic formulations. It demonstrates that, for the important special case of convex loss functions, both the dynamic and the linear programming formulations of optimum reservoir operation policies as developed over the past decade still have a long way to go to match a 55-year old mass-curve technique in terms of exactness, accuracy, as well as computational efficiency. Last but not least, it shows that the mass-curve technique provides insights into the problems of storage reservoir operation which are entirely out of reach of the systems-analytic methods and can significantly enhance the art of reservoir design and operation.

INTRODUCTION

The plot of a function $X(t)$ defined as an integral

$$X(t) = \int_{t_0}^{t} x(\tau)\, d\tau \tag{1}$$

and called the mass diagram or the mass curve has been used as a convenient tool for graphical integration in many engineering disciplines for well over a century [*Šolin*, 1872, 1874; *Iyanaga and Kawada*, 1977]. For example, in railway and road engineering it has been used for designing the grade line so that quantities of cut and fill would balance; in structural engineering for spacing beams so that they would carry specified proportions of total load; in hydraulic engineering for the design of surge tanks and spillways; in water resource engineering for the delimitation of boundaries between parts of a storage reservoir capacity allocated to different uses and for finding various relationships between reservoir storage capacity and reservoir yield.

As the term suggests, mass-curve methods were originally developed as graphical techniques though the mass curve was usually plotted with the aid of its numerically calculated ordinates. However, the graphical mode itself, while often facilitating elegant shortcuts on the way to a solution and always having a superior explicative potential, is not the essential feature of mass-curve methods, for they can be applied in either the graphical or the numerical mode. What is essential is the concept of employing an integral $X(t)$ of a function $x(t)$ rather than the function $x(t)$ itself. In this regard, mass-curve methods bear a similar relation to methods working with the original functions as, for instance, integral equations bear to algebraic equations.

Unfortunately, the century-old term with a distinct graphical bias may be at the root of a rather popular myth that mass-curve methods are essentially graphical techniques rendered obsolete by the advent of computer-based systems analysis. In the storage reservoir field, it has become almost a ritual to jump on the systems-analytic bandwagon from the springboard of mass-curve renunciation. This ritual has had the same purpose as the simple but effective promotion trick employed in bazaars of all ages and all parts of the world: to make a new gadget shine bright, it must be displayed against a dark background. However, the reason why in storage reservoir analysis mass-curve techniques were chosen as the most desirable background of this sort was not merely the term's lack of systems-analytic appeal. Much more significant was the fact that, by the time the systems-analytic bandwagon arrived on the American scene in the early 1960's, the term 'mass-curve technique' had already been considerably discredited by having acquired there at least two new and artificially restrictive meanings which are discussed in the next section. The rather loose semantic practice had done its job: the baby was thrown out with the bath water, or so, at least, it seemed.

But was it really so? In reality, mass-curve techniques were not, and could not have been, discarded in storage analysis, despite numerous claims to the contrary, the simple reason being that, as *Streiff* [1927] put it half a century ago, "... the [storage] depletion diagram is a mass-curve." What the systems-analytic cult had changed was mainly the façade: as in many other areas, previously simple concepts are now blurred by the use of pretentious language and largely ornamental mathematics and 'research results of painful modesty are introduced as solutions to problems of great depth and complexity' [*Berlinsky*, 1976]. Thus for instance, the storage depletion diagram has become the 'system's behavioral signature' and the mass-curve method for its computation is now referred to as the 'sequent peak algorithm,' often presented as something new and different [*Loucks*, 1976].

In this context it seems that, were it not for Hazen, systems analysis might have descended upon storage reservoir design half a century earlier than it did. For, when he was explaining the substance of his famous contribution to storage analysis [*Hazen*, 1914*a*], he observed [*Hazen*, 1914*b*]: '... it may be well to point out that the methods now proposed do not in any way supersede [the mass-curve method]. On the contrary, the mass-curve method is used as a basis for computing the storages on which all the tables and diagrams in the paper are based. Practically, however, the mass curves were not drawn, because it was found that the results might be obtained more easily arithmetically.'

This paper concentrates on the problem of reservoir performance (operation) optimization: one of the areas where systems analysis has been most aggressive and successful in promoting its image of unchallenged superiority, and where a mention of mass-curve methods is generally viewed as an

Paper number 8W0968.
0043-1397/79/008W-0968$01.00

ominous sign of advancing senility. In particular, it will be shown that after more than a decade of concentrated effort by an army of high-caliber researchers and after volumes of dissertations piled high and deep, swollen with jargon of Hegelian obscurity and bursting with a frightening overkill of equations that have exhausted several times over the notational capacity of two alphabets and caused breakdowns of many a computer, the pursuit of reservoir release optimization by the two most popular systems methods, linear programming and dynamic programming, is now slowly closing in on a solution equivalent to a result obtained by a more than 50-year old mass-curve method with the unpretentious name 'la règle du fil tendu,' or 'the stretched-thread rule.' This method was introduced by *Varlet* [1923] and has long been part of the standard repertoire of undergraduate dam-design courses at many technical universities throughout Europe. A corollary: solution of a simple policy-optimization problem requires about an hour or two by linear programming, a few minutes by dynamic programming, and a few hundredths of a second by the stretched-thread method despite the fact that the method is not well suited to computerization. Another corollary: while the linear programming and the dynamic programming solutions have a limited accuracy depending on the coarseness of the discrete representation of the storage and draft ranges, the stretched-thread method yields the solution with a maximum accuracy possible since no discretization is employed. One more corollary: while the time requirement for the dynamic programming solution grows with an increase in reservoir storage capacity, it decreases in the case of the stretched-thread method thus defying the 'curse of dimensionality.'

However, before proceeding to the main topic, a few words about the history of the mass-curve technique in storage reservoir analysis seem in order.

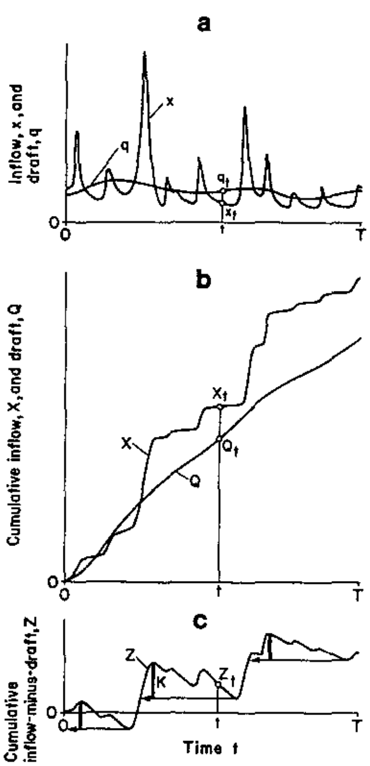

Fig. 1. Definition sketch for Rippl's mass curve method: (*a*) hydrographs of inflow x and draft q, (*b*) mass curves of inflow X and draft Q, and (*c*) residual inflow mass curve Z defined in Rippl's sense (2) and his procedure of determining the storage capacity K necessary for non-failure reservoir operation during the period T.

MAKING RIPPLES ON THE STAGNANT WATERS OF RIPPL'S MASS-CURVE ANALYSIS

Application of the mass-curve technique to storage reservoir analysis was introduced by *Rippl* [1883]. He plotted a residual mass curve Z of reservoir inflow x (represented by a given streamflow series) relative to reservoir draft q (represented by a given series of desired reservoir outflows),

$$Z_t = \int_0^t (x - q)\, d\tau = \int_0^t x\, d\tau - \int_0^t q\, d\tau = X_t - Q_t \qquad (2)$$

and used it to find the smallest storage capacity K necessary to supply the desired draft without failure throughout the whole period under consideration, T. This storage capacity is equal to the storage that would be depleted only in the most severe critical period. It was found by Rippl as the maximum of all the minimal critical-period fillups of a hypothetical semi-infinite (top-less) reservoir during the period T (Figure 1).

At the end of the nineteenth century, and independently in Europe (*Müller* [1914], referring to his earlier paper of 1896) and in America (*Tighe* [1914], referring to a report by John R. Freeman, published in 1900), Rippl's original method underwent two modifications. The first was the replacement of the residual mass curve Z by the inflow mass-curve X while replacing the horizontal tangents to Z shown in Figure 1*c* by tangents to X drawn parallel to Q; the second was the replacement of the 'critical-period fillup' concept with a concept of a 'critical-period depletion,' i.e., the substitution of a 'bottom-less' reservoir concept for Rippl's 'top-less' one. These changes had, of course, no effect on the result. (It may be mentioned that Rippl's concept, which assumes the reservoir initially empty, is more realistic, especially in the case of large reservoirs, than the modified concept which assumes it initially full.)

Whereas in Europe the development of mass-curve techniques further diversified and they were applied to solving many other problems besides the one posed by Rippl [*Kresnik*, 1897; *Jílek*, 1907; *Stupecký*, 1909; *Schoklitsch*, 1923; *Varlet*, 1923; *Novotný*, 1925, 1954; *Kritskiy and Menkel*, 1935, 1952; *Schaffernak*, 1935; *Ježdík*, 1937; *Ivanov*, 1946; *Hurst*, 1951; *Lyapichev*, 1955; *Kratochvíl*, 1961; *Votruba and Broža*, 1963; *Klemeš*, 1963, 1966; *White*, 1965; *Otnes and Raestad*, 1971], in America the development of mass-curve techniques and, indeed, of storage analysis, has virtually stopped and 'Rippl's method' (in fact, the modified Rippl's method) has become something of a petrified image of the mass-curve technique potential in storage analysis, perpetuated from one textbook to the next [*Mead*, 1950; *Linsley et al.*, 1949; *Babbit et al.*, 1955; *Linsley and Franzini*, 1972]. Thus in the American context, *Fiering*'s [1966] observation can be considered rather accurate: 'Rippl published his technique for mass-curve analysis in 1883, and the art of designing reservoirs was stagnant, except for a few innovations introduced by Allen Hazen and Charles Sudler until a few years ago.'

It is not surprising that after about 50 years of repetitious references to mass-curve analysis predominantly only in connection with Rippl's method, the two concepts have become virtually synonymous on the American scene and, in the usage of some authors, the mass-curve technique has acquired the two following restrictive meanings: (1) a technique for the determination of reservoir storage capacity for nonfailure operation under a given draft and for a given period of time; in other words, a technique for the determination of the range of cumulative departures of flow with respect to draft; (2) a technique using for the determination of reservoir storage (as described above) the historic flow record as a design period.

This 'semantic quirk' (M. B. Fiering, personal communication, 1978) has not arisen in Continental-European schools of storage analysis. In the first place, Rippl's paper (probably because of being published in English) has never acquired the prominence in continental Europe that it did in the English and especially the American schools. Second, as already indicated, mass-curve techniques have been used in continental Europe not only for the determination of storage for a given draft but also for other purposes, such as the determination of the draft or the flood reduction corresponding to a given storage, optimum flow-equalization, hydro-plant operation for a given power output, reservoir rule-curves, frequency and length of failure periods, amounts of deficits, etc.

As a rule, the above modifications of the original meaning of mass-curve analysis have not been explicitly identified and the semantic usage of the term has been quite loose, often changing back and forth between the literal and some of the symbolic meanings within one paper or book. Thus the reader of scientific literature on storage design and operation published over the past decade [e.g., *Fiering*, 1967; *Buras*, 1972; *Jackson*, 1975] may be puzzled by criticisms of 'mass-curve analysis' which are themselves backed by results derived by mass-curve analysis and by various contradictory claims and comparisons which do not seem to make sense.

Thus for example, *Fiering* [1967, p. 7] states that the 'mass diagram does not help the designer to establish or calculate the risk to be taken with regard to water shortages during periods of low flow' while later [*Fiering*, 1967, p. 9] he implies that the risk estimation by the mass diagram is possible when he says: 'If the mass curve is used to estimate the frequency and magnitude of flow deficiencies. . . .' One may easily overlook that, in the first statement, the term has the two symbolic meanings identified earlier (and is technically identical to 'Rippl's method'), whereas in the second statement it has the original literal meaning. Still later [*Fiering*, 1967, p. 9] the storage risk-analysis to be carried out in the book is referred to as a 'study of operational hydrology and mass curve analysis' where the latter term has been used in only the first of the two symbolic meanings as is evident from a more detailed description [*Fiering*, 1967, pp. 10, 11].

In another example [*Jackson*, 1975], mass-curve analysis is contrasted with the use of synthetic flow series in reservoir analysis. This would make no sense unless the concept of mass curve analysis were reduced to the second symbolic meaning, which Jackson is doing by definition: 'This technique uses only the historical flow sequence in evaluating proposed designs. . .'

From what has been said it may appear that the unfortunate semantic quirk could be cleared up and the dignity of mass-curve analysis restored if the term Rippl's method were used systematically, as it occasionally has been [*Fiering*, 1967; *Butcher and Fordham*, 1970], for the technique of finding the reservoir storage capacity for a given draft from the historic flow record. While this term would be technically correct, the resulting redirection of the current criticisms of 'mass-curve analysis' to Rippl's method would be a great injustice to Rippl, because it is the later use of Rippl's method, rather than Rippl's own use of it, to which the criticisms are relevant.

As an illustrative example, let us consider in detail 'the most fundamental criticism' [*Butcher and Fordham*, 1970] or one of the 'principal defects' [*Fiering*, 1967] of Rippl's method, i.e., the fact that it is based on historical records of streamflow. This criticism addresses the common practice of using the historical streamflow record as the sole model for the streamflow series during the design period despite the common knowledge that the historical record has virtually a zero probability of being duplicated in the future. The fact is that nowhere in Rippl's paper is such a practice recommended. Rippl's purpose was not to tackle the problem of what is the best flow record to be used for the design period but how to analyze such a known flow record correctly. His main purpose was to show that the then common practice of designing reservoir storage capacity to meet the demand in one year only, even the driest one, is unsafe since several mild droughts in series may require a larger storage capacity than any single drought, however severe. He demonstrated his thesis by analyzing storage requirements for a hypothetical demand and the historical rainfall record in Vienna, and was able to conclude: 'The records of the rainfall at Vienna prove this assertion.' For the given purpose, the mass-curve method which he used would even today be the best to employ. Rippl described it very precisely as a '. . . method of determining the capacity required for storage to equalize the supply and demand during any period for which rainfall observations are available.' The 37 years of rainfall observations at Vienna were available and he used them to demonstrate the method and to prove his point.

The closest Rippl came to the problem of an appropriate design period was in stating that 'First the supply [i.e., the inflow] to the reservoir and the outflow are estimated for successive equal periods of time, usually one month, and for the whole period of time to be considered'—which is exactly the objective we are still trying (with not much success) to achieve today. Rippl wisely refrained from giving advice as to how such an estimate should be made, although it can be assumed that he would have used the historical record. However, even if he had explicitly recommended the use of the historical record (as was to become a standard practice), what more sensible advice could anyone have given at that time? We should appreciate that it was 1883, when A. A. Markov, age 27, did not have the faintest idea about a Markovian process and when even the farsighted Jules Verne did not dream of computing machines that were to give rise to synthetic hydrologic series, thus 'making available' this somewhat questionable substitute for the historical records. There is no doubt whatsoever that Rippl and those who followed in his steps would gladly have used better estimates of future flows, had they been available, than those represented by the historical record.

To digress at this point, it may also be mentioned that historic streamflow records can still compete with synthetic series as design-period flow models for a number of storage-related problems. These include the determination of failure period frequency and deficits of a finite reservoir [*Kritskiy and Menkel*, 1935; *Klemeš*, 1966], the optimization of reservoir operation policies [*Jettmar and Young*, 1975], the determination of the seasonal storage component [*Klemeš*, 1963] and, in combination with some elementary probabilistic assumptions, even the estimation of the complete storage-draft-reliability relationship [*Lyapichev*, 1955; *Gould*, 1961; *White*, 1965; *Klemeš*, 1970].

To conclude the argument, it has to be emphasized (1) that the use of a historic record by Rippl was fully justified for the problem which he examined and (2) that Rippl's method as such is not in any way tied to the use of a historic record, its essence being the determination of the range of a residual mass curve. Those who contrast Rippl's method with new techniques of storage analysis are well advised to note that the supposedly modern 'sequent peak algorithm' is nothing else than an 'inverted' Rippl's method which could well be termed the 'sequent trough algorithm' [*Klemeš*, 1978]. And those who may be tempted to reject it in the name of systems analysis should not overlook the fact that, in the systems-analytic jargon, the method represents a 'backward-moving, forward-looking recursive sequential maximization' which is the basic mode of the dynamic programming technique. For Rippl's method can be formally written as [*Klemeš*, 1979].

$$K_i = \max(K_{i-1}, C_i) \qquad i = 1, 2, \cdots, n \tag{3}$$

the result being given as

$$K = K_n \tag{4}$$

where the subscript i runs backward in time, C_i is the minimal fillup necessary for the ith dry season, and K_i is the minimal storage capacity necessary for the period from the beginning of the ith dry season to the end of the whole period T.

Economic Interpretation of Physical Objectives Used in Storage Mass-Curve Analysis

There are two basic physical objectives that have been used in solving streamflow-regulation problems with the aid of mass-curve analysis: (1) regulation based on a firm value of target release ('safe draft') which is either constant or a periodic function of time and (2) regulation aimed at the greatest possible equalization of reservoir outflow. The second objective has often been regarded as the basic aim of streamflow control since the need for such control usually arises from a desire to mitigate losses caused by flows either too low or too high. Obviously, the combination of low flow augmentation and flood reduction results in a tendency to flow equalization. *Kritskiy and Menkel* [1952] consider such regulation as having a 'maximum economic effect' and *Kratochvil* [1961] calls it 'ideal.' Since, however, the maximum equalization of flow cannot be achieved without prior knowledge of the future inflows, 'rule curves' have been designed on the basis of the 'best' reservoir performance in a historical flow record in the hope that regulation based on such curves, even if not fully achieving the second objective stated above, will approach it more closely than regulation based on the first objective. Thus, in effect, the two objectives listed above have usually been regarded as the lower and upper limit of regulation optimality, respectively.

The older literature on streamflow regulation does not, to my knowledge, contain any explicit economically based proofs supporting this widely accepted view. However, it transpires that this view has developed in response to an implicit perception of the economic loss function as a convex function of streamflow. For example, when discussing losses incurred due to a reduction of draft below the target level, *Kritskiy and Menkel* [1952] say: 'Some reduction in water demand can always be achieved without a substantial loss, but a significant reduction of water supply to the consumer always causes a disproportionate increase in the loss.' In principle, the same philosophy applies in flood protection.

A closer examination reveals that, on the assumption of a convex loss function, regulation aimed at objective 1—the release of target draft (defined here as the value of flow at which the loss function has its minimum)—represents the optimal regulation under conditions of extreme hydrologic and/or economic uncertainty regarding the future [*Klemeš*, 1977*b*]. Likewise, it can be shown that regulation aimed at objective 2—the maximum equalization of reservoir outflow—is optimal under conditions of perfect knowledge of future streamflow combined with minimum economic uncertainty (zero discount rate risk premium). The common view that flow equalization is the best operating policy is held so widely probably because it is based on sound economic intuition gradually acquired by man who, through his dramatic historical experiences with floods and droughts, over countless generations, has learned the economics of streamflow control the hard way.

Before proceeding to show that the current dynamic and linear programming formulations of the performance optimization problem for a finite reservoir lead to flow equalization, let us demonstrate the optimality of an equalized outflow, given a convex loss function, for an infinite (unconstrained) reservoir.

Any stationary regulation scheme will result in some distribution of outflow y with mean $E[y]$ and variance Var $[y]$. A regulation scheme aimed at flow equalization would ideally result in a constant outflow equal to $E[y]$. Considering a loss function $L(y) = y^a$, which is convex for $a < 0$ and $a > 1$ and concave for $0 < a < 1$, we shall examine the magnitude of the average loss $E[L(y)] = E[y^a]$ corresponding to the variable outflow y as compared to the loss $L(E[y]) = (E[y])^a$ corresponding to the equalized outflow $E[y]$. For this purpose we express $E[y^a]$ in terms of $E[y]$ by expanding y^a in the neighborhood of $E[y]$ by the Taylor series. Retaining the first three terms of the expansion and rearranging we obtain [*Ventsel*, 1964]

$$E[y^a] = (E[y])^a + \tfrac{1}{2}a(a-1)(E[y])^{a-2} \operatorname{Var}[y] \tag{5a}$$

Since $E[y]$ and Var $[y]$ are positive, it follows that:

1. A convex loss function ($a < 0$ or $a > 1$) yields

$$E[y^a] > (E[y])^a \tag{5b}$$

thus rendering the equalized outflow economically superior to a variable outflow.

2. A concave loss function ($0 < a < 1$) yields

$$E[y^a] < (E[y])^a \tag{5c}$$

indicating that any outflow variability is better than its constancy thus implying that no general optimization is possible in this case.

3. A linear as well as a constant loss function ($a = 1, a = 0$) both yield

$$E[y^a] = (E[y])^a \tag{5d}$$

which means that the overall economic effect is independent of the outflow pattern.

Although the employed simple form of the loss function does not exhaust all the possibilities that can be encountered in practice, and the results have only an approximate validity because of the absence in (5*a*) of the remainder of the expansion, they nevertheless indicate a fundamental soundness of the old empirical wisdom that the purpose of reservoirs is to equalize the streamflow.

From Dynamic Programming Optimization to the Method of Stretched Thread

Since it was published a decade ago, *Young*'s [1967] paper has become a classic in the dynamic programming school of reservoir performance optimization. The essence of Young's approach is the application of a deterministic dynamic programming procedure to the problem of finding an optimum outflow series from a reservoir subject to a given series of inflows, and the definition of an optimum reservoir release as a statistical linear function of the current value of storage and the current (or the current and the forecast immediately following) inflow, a relationship derived by regression analysis from the optimized outflow series. His technique, known either as Monte Carlo dynamic programming or implicit stochastic dynamic programming, has the virtue that it can be applied in conjunction with any type of streamflow model including the historical flow record [*Jettmar and Young*, 1975].

Fiering, under whose guidance Young developed his technique, was the first to see the technique's potential and pointed out that Young proved '. . . several surprising theorems and generalizations. . .' [*Fiering*, 1967]. However, perhaps the most surprising aspect of Young's results seems to have gone unnoticed at that time: embedded in Young's derivations is a proof that the method he employed for finding the optimum reservoir outflow is identical to *Varlet*'s [1923] mass-curve method illustrated in Figure 2.

To substantiate this claim it is sufficient to retrace the first part of Young's analysis concerned with the derivation by calculus of variations of an optimal outflow from an unconstrained reservoir. The second part dealing with the development of a dynamic programming procedure will be found redundant since it represents nothing more than a stepwise approximate solution to the same problem in the more complicated case of a constrained reservoir where a direct exact solution was considered as being out of reach.

First we note that what Young calls cumulative inflows and defines as a monotonic continuous nondecreasing and twice differentiable function of time is the inflow mass curve X; likewise, his cumulative draft function (in an unconstrained reservoir the outflow is equal to the draft) is the outflow mass curve Y, and his storage function S is the storage depletion diagram which, in an unconstrained reservoir, is identical with the residual mass curve Z defined in Rippl's sense (Figure 1). The double differentiability of X and Y, and thus also of S, implies that the rates of both inflow and outflow vary in a continuous fashion; following Young, the respective derivatives (all taken with respect to time) will be denoted as X', X'', Y', Y'', S', S'', (note that X' and Y' are the inflow and outflow rates designated by x and $y = q$, respectively, in Figure 1). The loss function will be denoted as $L = L(Y')$; Young assume it to be twice differentiable with respect to the outflow rate and writes $L' = dL/dY'$ and $L'' = d^2L/dY'^2$.

Young formulated the optimization problem as a variational problem of finding S as a function of X such that the loss accumulated over a period T is minimized; from this condition he then derived the optimum outflow rate Y'. Here, we slightly simplify this formulation by posing the problem as finding directly the optimum outflow mass curve Y as a function of X. Thus the condition of optimality is written as the minimization of a functional

$$F[Y] = \int_0^T L(Y')\,dt \tag{6}$$

In order to be a solution of this variational problem, the outflow mass curve Y must satisfy the Euler-Lagrange differential equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial Y'}\right) - \frac{\partial L}{\partial Y} = 0 \tag{7}$$

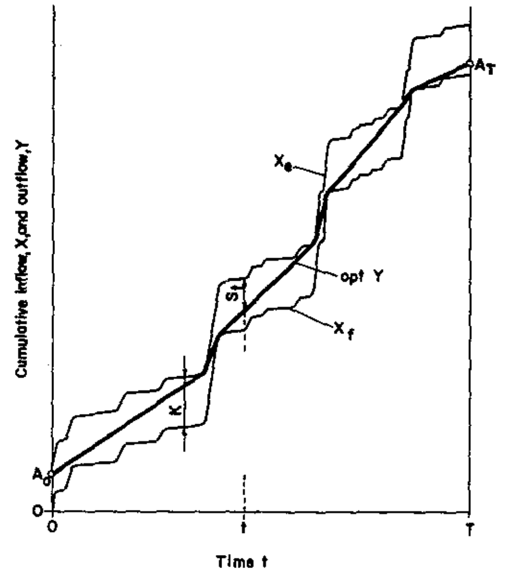

Fig. 2. Mass curve of optimum reservoir outflow opt Y obtained by the stretched-thread method and represented by the shortest path between two points A_0 and A_T at the opposite ends of a corridor formed by two mass curves of reservoir inflow X_e and X_f displaced by reservoir storage capacity K. Curve X_e represents the line of empty reservoir and X_f the line of full reservoir since the instantaneous value of reservoir storage S_t is given as the vertical distance between X_e and opt Y at time t.

Since the loss is a function of the outflow rate only (it does not depend on the accumulated outflow) we have

$$\partial L/\partial Y = 0 \tag{8}$$

$$\partial L/\partial Y' = L' \tag{9}$$

and, since the outflow rate is a function of time, (7) becomes

$$L''Y'' = 0 \tag{10}$$

Young assumes that $L'' \neq 0$ whence

$$Y'' = 0 \tag{11}$$

which has a general solution

$$Y = c_1 t + c_2 \tag{12}$$

After the introduction of the initial condition $Y_0 = 0$ and the end condition $Y_T = X_T + S_0 - S_T$ into (12) the resulting optimum outflow mass curve is obtained in the form

$$Y = \frac{1}{T}(X_T + S_0 - S_T)t \tag{13}$$

with the corresponding optimum outflow rate

$$Y' = (X_T + S_0 - S_T)/T \tag{14}$$

For the stationary case, the initial and terminal storages become equal and, since X_T/T is the mean inflow during the period T, the outflow mass curve can be written as

$$Y = \bar{x}_T t \tag{15}$$

the optimum outflow rate being

$$Y' = \bar{x}_T \tag{16}$$

The only required modification to these results of Young is a restriction to their validity. Whereas Young assumed that they are valid for both convex and concave loss functions because

$L'' \neq 0$ holds for both cases, they are actually valid for convex loss functions only. The reason for this restriction follows from the fact that the Euler-Lagrange equation, while being a necessary condition, is not a sufficient condition. Indeed, it is considered doubtful that a sufficient condition for the absolute minimum of $F[Y]$ can be found in a general case '. . . since the process of eliminating possible solutions by the stipulation of additional necessary conditions has no obvious termination' [*Dreyfus*, 1965]. It takes, however, only one more step on the endless ladder of necessary conditions to eliminate concave loss functions from candidacy for admissibility. This is the introduction of the strengthened Legendre condition [*Dreyfus*, 1965] stipulating that

$$\frac{\partial^2 L}{\partial Y'^2} > 0 \tag{17}$$

and satisfied by a convex function only.

Returning now to the above derivation of the optimum outflow, we see that (13) contains a proof that Varlet's stretched-thread method is nothing else but an exact variational solution to our optimization problem for a constrained reservoir. The argument is as follows. For an unconstrained reservoir, the equation establishes that the mass curve of the optimum outflow is a straight line, i.e., the shortest path, connecting the points A_0 ($t = 0$, $Y_0 = 0$) and A_T ($t = T$, $Y_T = X_T + S_0 - S_T$), S_0 and S_T being the initial and terminal storages in the period T, respectively. (This is equivalent to a statement, following from (14), that a uniform distribution of the total amount of water available in a given period T, i.e., an 'equalization of the water supply,' is the optimum release policy.) If reservoir constraints on storage, $S_{\min} = 0$ and $S_{\max} = K$, are interpreted in Varlet's sense as 'lines of empty and full reservoir' [see also *Schoklitsch*, 1923], an unconstrained reservoir is any constrained reservoir for which the outflow mass curve is not interfered with by these lines (Figure 3*a*). Thus (13) is also valid for an 'equivalent constrained' reservoir with storage capacity $K \geq \min K$, starting storage $S_0 \geq \min S_0$, and terminal storage $S_T \geq \min S_T$ (Figure 3*a*). To use Varlet's terminology, the smallest storage capacity of an equivalent constrained reservoir can be obtained by moving the lines of empty and full reservoir as close to each other as possible without bending a thread stretched between the terminal points A_0 and A_T. By moving the two lines still closer together,

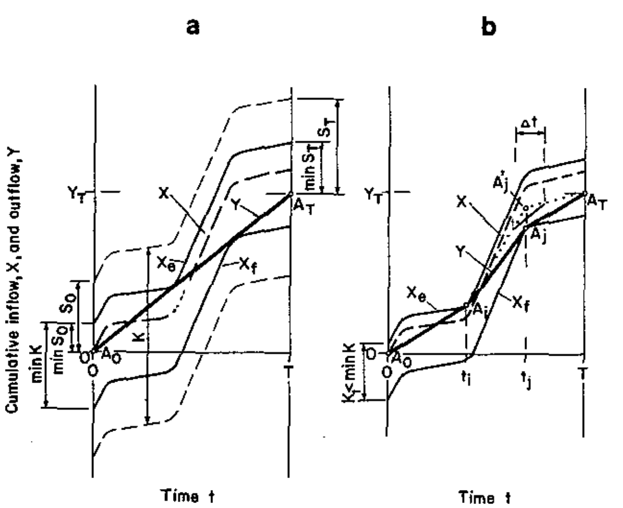

Fig. 3. Definition sketch for an interpretation of the stretched-thread method as the solution by calculus of variations of the outflow optimization problem for a constrained reservoir.

say to a distance $K_1 < \min K$, the thread would be bent at certain points, in our case A_i and A_j with time coordinates t_i and t_j (Figure 3*b*), thus violating the validity of (13) as well as the initial assumption regarding the existence of Y'' in the whole interval $(0, T)$. However, if we now consider the three segments of the 'broken' or 'bent' stretched thread separately we see that for each of them the storage capacity K_1 can be regarded as the minimum capacity of an equivalent unconstrained reservoir. That is to say that the validity of (13) has been restored piecewise within time intervals corresponding to the individual segments of the stretched thread. Thus these segments represent mass curves of piecewise optimum outflow for a reservoir of a storage capacity K_1. It can be easily verified that, if the segments had other terminal points than A_i and A_j, there would always be intervals Δt within $(0, T)$ in which the broken line formed by the segments would not represent the optimal outflow since it would be possible to replace it there with a straight line without violating the storage constraints (e.g., see the case indicated by a dotted line in Figure 3*b*, in which point A_j has been replaced by A_j').

In the language of the calculus of variations, the problem is one of finding the shortest path within the neighborhood of X bounded by the lines X_e and X_f. *Dreyfus* [1965] makes the following comment on this problem:

> When we presented the differentiated form of the Euler-Lagrange equation, we noted that this equation was to hold between points of discontinuity in the derivative of the solution function, and that the points where the solution function has a discontinuous derivative are called corners. The complete solution may consist of segments, each satisfying Euler-Lagrange equation between corners. For the shortest-distance problem, the Euler-Lagrange equation implies that minimizing curves are straight lines between corners.

Obviously, Varlet's method of stretched thread is a graphical method for the identification of the corner points referred to by Dreyfus, the stretched thread, itself representing the solution function.

FROM LINEAR PROGRAMMING OPTIMIZATION TO THE METHOD OF STRETCHED THREAD

The mainstream of the linear-programming efforts in reservoir optimization is represented by the development of the so called chance-constrained linear decision rules (LDR) by ReVelle and his coworkers. It started in 1969 [*ReVelle et al.*, 1969] and has since reached the proportion of a respectable saga of five parts, the last being by *Gundelach and ReVelle* [1975]. A variation on the theme of the fourth installment [*ReVelle and Gundelach*, 1975] has recently been added by *Luthra and Arora* [1976].

The aim of these efforts has been optimization of a physical rather than an economic objective. Specifically, the objective has been to minimize the reservoir storage capacity required for operation under the following conditions: (1) the release must exceed a given minimum flow with a specified probability, (2) the release must not exceed a given maximum flow with a specified probability, (3) the storage must not fall below a given minimum (higher than zero) with a specified probability, and (4) the storage must not rise above a given maximum (lower than the total storage capacity) with a specified probability. Apart from the minimum storage capacity required for such operation, the solution also specifies the LDR with the aid of which a release commitment at time t for the next time interval t, $t + \Delta t$ (i.e., with a slight simplification, the outflow during Δt) can be specified.

Before proceeding to show that the development of the problem has been moving in the direction of the stretched-thread method, it may be useful to clarify the meaning of the above mentioned chance constraints. According to the authors [*ReVelle et al.*, 1969] these constraints are incorporated into the analysis in the form of their so called 'certainty equivalents.' For instance, for a given month of the year the first of the four chance constraints is accommodated by taking as the chance-constrained release the value of natural flow whose probability of nonexceedance in that month is equal to the specified 'chance constraint,' e.g., 5%. The authors believe that this procedure amounts to an explicit solution of the optimization problem for the risk level so specified and, though admitting that these constraints '. . . do not imply guarantees about the storages or flows. . . ,' they claim that 'Reservoirs operated under linear rules based on chance-constrained formulations should perform at the levels of certainty that the designer or decision maker specifies.' However, *Loucks and Dorfman* [1975] have found that chance-constrained models of this category give conservative results. They observed: 'More active storage capacity was specified by the solution of the chance-constrained models than was actually needed to meet the reliability requirements defined by the chance constraints.' As one of the principal reasons for this discord, Loucks and Dorfman cite the fact that the above procedure implies the assumption of the simultaneous occurrence of the critical flows and/or storages in each period, an event whose joint probability is negligible. Moreover, such joint probability varies with the stochastic structure of the streamflow model so that the belief of *ReVelle et al.* [1969] that they had avoided the necessity for a trial-and-error approach in finding storage capacity or draft rate for a given risk of failure was not warranted. The chance constraints in the model of *ReVelle et al.* [1969] can only be regarded as rough initial estimates in a trial-and-error search, i.e., as deterministic constraints whose true probabilities have to be determined. A recent model by *Houck* [1978] seems to approach the explicit solution much better than do the previous models. With this issue clarified we can now return to the main topic.

The original release rule [*ReVelle et al.*, 1969] relied completely on the end-of-period storage S_{i-1} for the computation of release commitment for the ith period, $Q_i = \int_{t_{i-1}}^{t_i} q\, d\tau$, which was defined as

$$Q_i = S_{i-1} + b_i \qquad (18)$$

Here b_i is the decision constant to be determined; it represents a release commitment on account of the future inflow $X_i = \int_{t_{i-1}}^{t_i} x\, d\tau$ expected in the ith period, so that the terminal storage in this period will be $S_i = X_i - b_i$. Since the same holds for all end-of-period storages, (18) can be written as

$$Q_i = X_{i-1} - b_{i-1} + b_i \qquad (19)$$

This release rule was eventually replaced with a rule

$$Q_i = a_iX_i + a_{i-1}X_{i-1} + a_{i-2}X_{i-2} + \cdots - b_{i-1} + b_i \qquad (20)$$

thus making the release a weighted sum of the past inflows [*ReVelle and Gundelach*, 1975]. The weights $a_i, a_{i-1}, \cdots$ must be determined by some additional condition and the authors have proposed to use the condition of a minimum variance of the releases Q_i. As shown earlier by *ReVelle and Kirby* [1970], the minimization of the variance of reservoir outflows Y_i over a period T is equivalent to the minimization of the total loss over this period on the assumption of a quadratic loss function $L(Y_i) = \omega(Y_i - Q_0)^2$ where Q_0 is the target draft (i.e., a draft causing the smallest loss). This is so for the following reason. The minimization of the total loss can be replaced by the minimization of the average loss

$$E[L] = E[\omega(Y_i - Q)_0^2] = \omega(E[Y_i^2] - 2Q_0E[Y_i] + Q_0^2) \qquad (21)$$

and since $\text{Var}\,[Y_i] = E[Y_i^2] - \{E[Y_i]\}^2$, the average loss can be expressed as

$$E[L] = \omega\ \text{Var}\,[Y_i] + \omega(E[Y_i] - Q_0)^2 \qquad (22)$$

The average outflow is fixed since the total outflow is always equal to the total inflow plus the change in storage, so that the only term in (22) subject to minimization is $\text{Var}\,[Y_i]$ which can be replaced by $\text{Var}\,[Q_i]$ since outflows should ideally be equal to drafts ((22) also shows that the best possible loss function is one in which the target draft is equal to the mean inflow since $E[Y_i] = E[X_i]$).

Obviously, the absolute minimum for the variance of Y_i is zero and corresponds to the case where all outflows are equal to the average outflow, a situation achievable in an unconstrained reservoir or in an 'equivalent constrained reservoir' as defined in the preceding section. If the reservoir has fixed constraints (whether rigid, like the empty and full reservoir, or flexible like the freeboard or a minimum storage limit) the averaging of outflow can only be achieved piecewise within subperiods in which the operation of the constrained reservoir can be made similar to the operation of an unconstrained reservoir. The best combination of such subperiods and their respective optimum outflows is the one obtained by the stretched-thread method, since any departure of the outflow mass-curve from the shortest path through the corridor between the constraining lines would either increase the outflow variance or violate the constraints. This is true both for the variance of all outflows during the design period and for the variances of outflows in individual months (or seasons) of the year as the condition was originally posed by *ReVelle and Gundelach* [1975].

Hence the original problem as defined at the beginning of this section can be solved with the aid of the mass curve technique. The method of solution, consisting in the development of reservoir-operation charts, has been described in detail by *Kritskiy and Menkel* [1952], *Lyapichev* [1955], *Votruba and Broža* [1963], and others. An example of the method as applied to flow regulation for hydropower generation is shown in *Kratochvil* [1961] and an application for irrigation water supply in *Klemeš and Hejl* [1962].

The basic principle of this method is as follows. First the two minimum storage capacities necessary for the desired minimum and maximum outflows are found by drawing, respectively, the upper and lower tangents (with appropriate minimum and maximum slopes) to the inflow mass curve. The larger of these two values, adjusted to accommodate the initial conditions, is taken as an estimate of the required storage capacity K for which the optimum outflows are found using the stretched-thread method. The family of storage depletion diagrams for all individual years of the design period, corresponding to this theoretically optimal operation, serve as a basis for specifying the freeboard and minimum storage values of various exceedance probabilities throughout the annual cycle and for selecting a 'rule curve' of a specified frequency level. The actual frequency of violation of the design constraints is then assessed by trial computations in which the rule curve is followed as long as $q_{\min} \le y \le q_{\max}$, the limiting releases q_{max} and q_{min} being applied within the freeboard and the minimum-storage violation zones, respectively. The desired solution is arrived at by trial-and-error.

ReVelle et al. [1969] suggested that 'Computers and mathematical optimization notwithstanding, . . . reservoir design is a creative art and . . . the quality of the design depends in great part on the designer's ability to visualize the interaction of all components of the proposed system.' However, in view of this belief, fully shared by the present author, it is surprising that *ReVelle et al.* [1969] regarded the linear programming formulation as an aid that can help the designer in this respect.

The extreme difficulty in providing insights into the problem by the linear programming approach is evident from the history of the LDR development. It took the authors 6 years to proceed from the original form of the release rule as given by (19) to the form given by (20). And yet, even this last form still remains far behind the exact variational solution of Varlet, not to mention the fact that the analytical form of (20) offers no useful insight whatsoever into the mechanism of reservoir operation save the rather trivial assertion that the present release may depend on the past inflows. It does not suggest how far into the past the dependence may extend, how fast it may decay, whether it decays at all, whether the reservoir size may have any influence on it, etc. All the 'meat' has to be filled in through the condition Var $[Q_t]$ = min, which, simple as it is, in the linear-programming formulation of *ReVelle and Gundelach* [1975] leads through a jungle of partial differential equations to a monstrous system of linear equations. In the light of the last quotation from *ReVelle et al.* [1969], the scheme is simply self-defeating; the designer's ability to visualize anything at all has been wiped out and he has to accept the computer output as something like a divine revelation.

For comparison, the following properties of the optimal release policy are immediately obvious from the stretched-thread method:

1. The optimal release at any time depends not only on the past inflows but also on the future ones.

2. It depends directly only on the immediate past and future flows (between the nearest corner points), the dependence on flows in the remote past and future being limited to the location of the corner points; hence follow the utilities of both the historical record and flow forecasting.

3. The optimal release does not depend on the current value of reservoir storage but only on the storage values corresponding to the two closest corner points.

4. These two storage values can only be either $S = 0$ or $S = K$ (or the values corresponding to the freeboard and the minimum storage constraints if specified).

5. Because of the preceding condition, the optimum outflow within the period Δt between two successive corner points can assume only the following three different values, all related to the mean inflow over Δt:

$$y_{opt}(\Delta t) = E[x]_t^{t+\Delta t}$$

$$y_{opt}(\Delta t) = E[x]_t^{t+\Delta t} + K/\Delta t$$

$$y_{opt}(\Delta t) = E[x]_t^{t+\Delta t} - K/\Delta t$$

These values would be properly modified if other than the absolute storage constraints were specified;

6. The increase of the distances between corner points with the increase of reservoir storage capacity makes the estimation of their location progressively more difficult as the storage capacity grows. This means that the difficulty in specifying the optimal release increases with reservoir size.

7. The latter difficulty is mitigated by the fact that with an increasing storage capacity the optimum releases are progressively better approximated by the value of the long-term mean inflow.

There are many possibilities for incorporating these properties into release rules. The inevitable uncertainty of future inflows makes the decisive parameters (distance between corner points, reservoir state at a corner point, position of the last corner point, average flow between corner points, etc.) random variables whose properties can be estimated from historic flow records and built into the rules in a probabilistic fashion with varying degrees of sophistication. It should be noted that most of the random variables to be considered are variables characterizing the residual mass curve of inflow. The large sampling errors of such variables are well known which points to difficulties to be expected with statistical optimization of reservoir performance, especially when applied to operation of a specific individual reservoir [*Klemeš*, 1977*b*].

COMPUTATIONAL EFFICIENCY OF THE STRETCHED-THREAD METHOD

An analysis of computational effectiveness of the dynamic and the linear programming techniques of reservoir optimization was done by *Gablinger and Loucks* [1970]. They found that, for an explicit stochastic formulation, the dynamic programming procedure requires about 1/20 of the computer time required for the linear programming procedure. For an assessment of computational efficiency of the stretched-thread method, a deterministic optimization of outflows from a single reservoir has been carried out for a 25-year series of mean monthly flows and a quadratic loss function, using (1) *Young*'s [1967] forward-moving dynamic programming algorithm and (2) a computerized algorithm for the stretched-thread method (appendix). The optimization was performed for several values of reservoir storage capacity. To ensure an approximately equal accuracy of results obtained by dynamic programming for different storage capacities K the same storage interval ΔS has been used for all values of K [*Klemeš*, 1977*a*]. This storage interval was derived from the condition that the computer storage available (a CDC 6000-CYBER 74 system was used) be fully utilized for the analysis of the largest reservoir storage capacity analyzed, K_{max}. The resulting number of reservoir states for this K_{max} was 201 yielding a ΔS which ensured that the accuracy of the computed optimum outflows was within 5% of the mean inflow, i.e., $y_{opt} - 0.05\bar{x} \leq y \leq y_{opt} + 0.05\bar{x}$. The solution using the stretched-thread method gives, of course, the exact values of optimal outflows. Figure 4 shows the computer time required for the computation of optimum outflows as a function of reservoir storage capacity for both methods. Figure 5 shows the stretched-thread method solutions displayed with the aid of residual mass curves (defined with respect to the mean inflow) for three values of storage capacity.

RESERVOIR PERFORMANCE OPTIMIZATION IN LIGHT OF THE STRETCHED-THREAD METHOD

It has been customary during the past decade to contrast storage mass-curve analysis with the modern systems approach to storage problems and to expose the 'defects' of the former in the light of the 'virtues' of the latter. Let us now reverse the point of view and look at the performance of two most popular representatives of the systems approach, the dynamic and the linear programming techniques, in the light of the results obtained by Varlet's 55-year-old mass-curve method.

The basic conclusion from the analysis presented on the preceding pages is that, for a deterministic formulation of the problem of finding an optimum reservoir operation policy (given a convex loss function or a concave gain function), Varlet's solution is the best: it represents a limit to which both the dynamic and the linear programming solutions converge. Dynamic programming approaches it by refining the discrete representation of the continuous variables of storage and time, linear programming by refining the discrete representation of time. Discretization of time usually has a natural finite limit since the inflow data are invariably represented by a discrete time series. Thus the linear programming technique can give the same result as Varlet's technique for the same inflow series while the dynamic programming technique cannot because of the additional necessity of storage discretization. However, both techniques soon encounter the 'curse of dimensionality': dynamic programming on account of the number of state variables, linear programming because of the number of equations to be solved. The less than maximum numerical accuracy as such is usually not a serious practical problem since even a numerically exact result can only be regarded as an estimate if for no other reason then because of the inherent uncertainties in the input data alone. What is a problem is that both methods put both analysis and result well out of reach of any direct scrutiny by any 'back-of-the-envelope' computations: one is left with little choice but to believe the programmer and, above all, the computer as *Fiering* [1976] has pointed out. How serious the consequences of this may be has recently been shown by *Klemeš* [1977*a*].

The piecewise constancy, independence of current storage, and abrupt changes at corner points of the optimum release in any specific situation should be sufficient to undermine our belief in the possibility of approaching the truly optimal release rules in the absence of foreknowledge of future flows since no systems-analytical or other legerdemain can produce the needed but unavailable information. Perhaps the main practical virtue of the stretched-thread method is that it helps us to understand this unpleasant reality by displaying the mechanism of optimal reservoir operation in its most naked form stripped of the camouflage of spurious mathematics and technical jargon. It demonstrates in the clearest possible way the importance of flow forecasting as well as the value of the historic flow record. It shows why small reservoirs need forecasts with shorter lead times than large reservoirs, demonstrates the operational robustness of a large reservoir in which the mean flow is the safest bet for optimal release, and makes it clear why rule-curve based operation policies can be quite reasonable for reservoirs for annual regulation but have hardly anything to offer in case of large reservoirs intended for long-term regulation. It explains why it is not so much the inflow rates that are important for optimum operation but rather the total inflow volumes for the reservoir 'working cycles': it is these volumes which determine the location of the corner points which are crucial for the determination of optimum outflows.

As usual, however, bonuses do not come free. The simplicity and elegance of Varlet's solution are paid for by its applicability to only a rather special optimization problem characterized by zero discount rate of future benefits and a simple form of a univariate loss function. On the other hand, the computational clumsiness and lack of clear intuitive meaning of the linear and dynamic programming approaches are compensated for by their greater generality and flexibility.

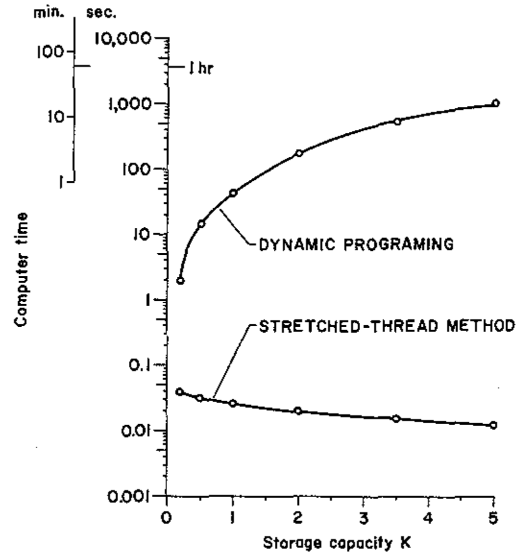

Fig. 4. Comparison of computational efficiency of the dynamic programming and the stretched-thread method solutions of reservoir outflow optimization. The indicated time is the central processing time required on a Control Data CYBER 74 system for outflow optimization in a 25-year series of mean monthly flows. The reservoir storage capacity is given in thousands of monthly inflow-rate residuals (conversion to m^3 is given as $K[m^3] = 2.6298\ K \times 10^9$). The error of the optimized outflow is $\pm 5\%$ of $\bar{x}$ for the dynamic programming solution, and zero for the stretched-thread method.

The point here has been not to call for purging the art of reservoir design of these techniques but bring them from their pedestal of false nobility down to earth by pointing out their intrinsic relationship to the traditional engineering technique of storage mass-curve analysis. A realization of such a relationship, it is hoped, may benefit the art of reservoir design by calming down the systems-analysis zealot on the one hand and by restoring the self-confidence of the engineer on the other.

CONCLUDING REMARKS

The aim of this paper has been to put storage mass-curve analysis into a proper perspective (and clear out some undesirable semantic underbrush in the process) rather than to criticize the dynamic and linear programming techniques whose usefulness and problem-solving power need no defense. However, since it was under the banner of an unchallengeable conceptual superiority of these techniques that the mass-curve analysis has often been attacked, this superiority had to be challenged in order that a balance could be restored. In this connection, many names had to be mentioned along with the criticized concepts to satisfy the current standards of source referencing. This, in a way, is rather unfortunate since it tends to divert attention from issues to persons, thus distorting the author's intent. To those who might be tempted to make inferences from this aspect of the paper, the following quotation is directed:

> We do not think any less of the profound concept of General Relativity in Einstein because the details of his formulation at this moment seem doubtful. For in science, as in literature, the style of a great man is the stamp of his mind, and makes even his mistakes a challenge which is part of the march of its subject. Science at last respects the scientist more than his theories; for by its nature it must prize the search above the discovery, and the thinking (and with it the thinker) above the thought.

[*Bronowski*, 1972].

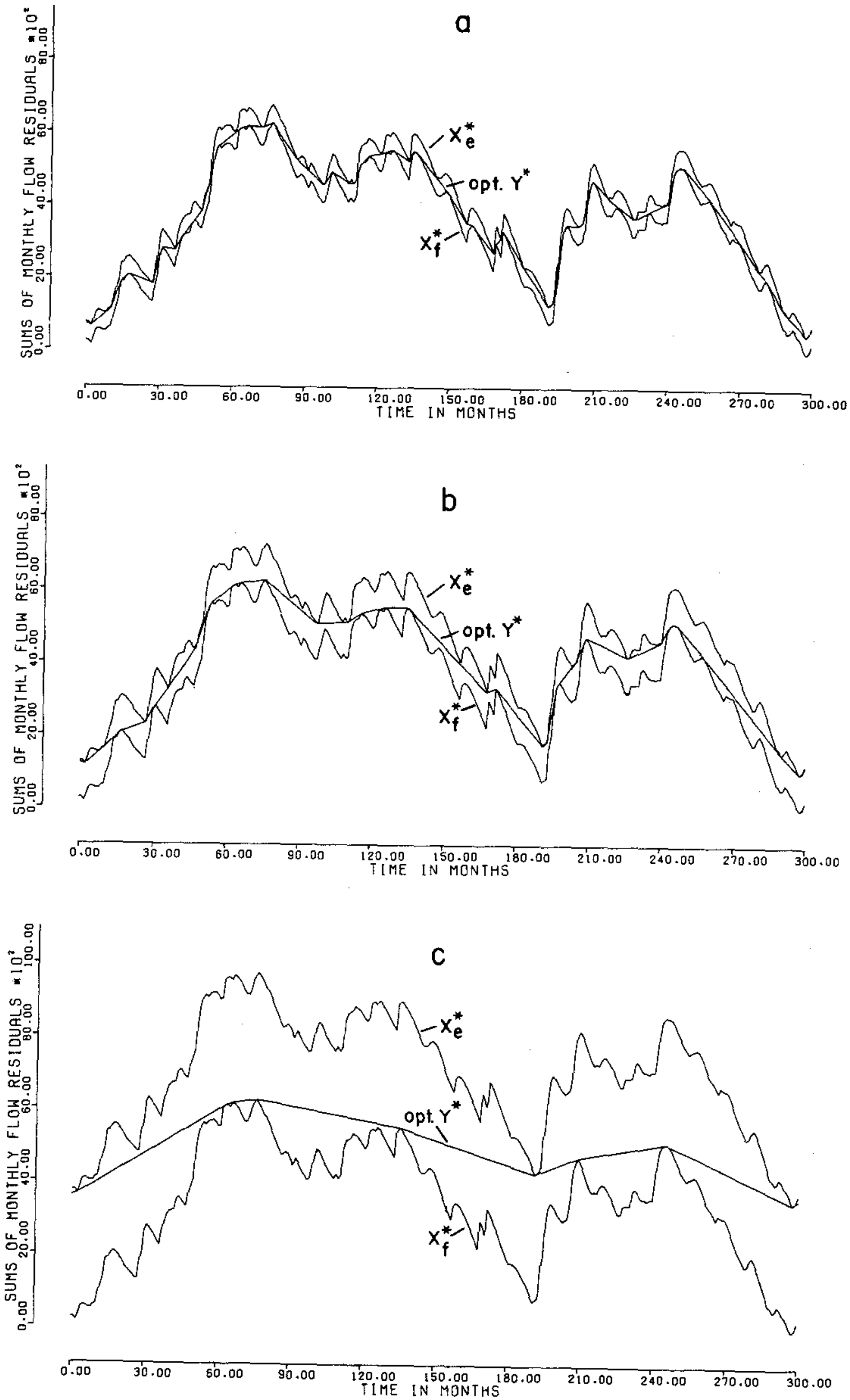

Fig. 5. Examples of the application of the stretched-thread method in conjunction with a residual mass curve of mean monthly inflows. The line of the stretched thread (the shortest path through the corridor X_e, X_f) represents the residual mass curve of the optimal outflow. (*a*) Storage capacity $K = 0.5$, (*b*) $K = 1$, (*c*) $K = 3.5$ (storage units same as in Figure 4).

Another aim of this paper has been to increase the awareness of the danger which the computer and the systems-analysis techniques pose, especially to the enthusiastic novice who has not yet had an opportunity to develop antibodies against the two highly contagious viruses, systems analysis snobbery and zealotry, the first identifying scientific approach with a blank contempt for conventional wisdom, the second identifying progress with computerization and with making simple things incomprehensible. Recently, *Berlinsky* [1976] has launched a frontal attack on these plagues of modern science. While one may not be willing to go with him all the way in dismissing most of systems analysis as a sham, a 'bouncy and outrageous attempt' to cover up trivialities and a lack of substance by ornamental use of mathematics and computers, one can hardly dispute the following observations made by *Fiering* [1976]:

> The engineering literature is replete with mathematical models, optimization techniques, Bayesian analyses, exotic formulations for synthetic flows, and all manner of computer studies. We seek optimal plans, optimal operating policies, optimal estimates of parameters, optimal anything. We are swept up in a litany of automatic computation, sensitivity analysis and model-making. It has become a new religion... And in the currently popular wave of condemnation of traditional engineering techniques... we can frequently lose sight of the fact that conventional wisdom might be selecting nonoptimal but significantly more robust results than our finely-tuned but brittle mathematical models...

In the context of the art of reservoir design this suggests that what a decade ago may have looked as stagnant waters may well be a reservoir of conventional wisdom and that the eager varlets of the computer should guard against making only hazes and ripples on the surface of this reservoir filled by Varlets, Hazens, and Rippls.

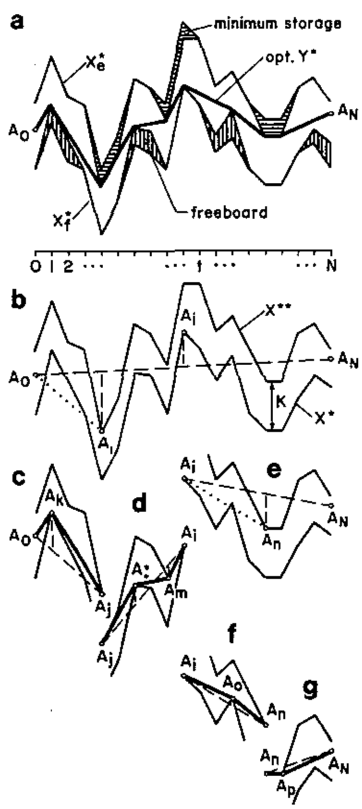

Fig. 6. Illustration of the development of a numerical procedure (computer algorithm) for finding the shortest path through an irregular corridor between two broken lines consisting of straight-line segments.

TABLE 1. Order of Corner Point Identification in Figure 6

Step	Corner point									
a*	A_0									A_N
b			A_j			A_i				
c		A_k								
d				A_l	A_m					
e								A_n		
f							A_o			
g									A_p	
Result	A_0	A_k	A_j	A_l	A_m	A_i	A_o	A_n	A_p	A_N

*Both terminal points are chosen arbitrarily between the relevant storage constraints at times $t = 0$ and $t = N$.

APPENDIX: PROCEDURE FOR NUMERICAL COMPUTATION OF SHORTEST PATH THROUGH A CORRIDOR OF IRREGULAR SHAPE

The procedure described here has been used in connection with the problem displayed in Figure 5 and is applicable when the corridor is enclosed within two broken lines consisting of straight-line segments and the end-points of the shortest path are specified. The lower broken line is given as a series $\{X_t^*\}$, $t = 0, 1, \cdots, N$, the upper broken line as $\{X_t^{**}\} = \{X_t^*\} + \{K_t\}$, where the series $\{K_t\}$ is the width of the corridor so that $K_t > 0$ for all t. In the storage reservoir problem, a variable width of the corridor arises if constraints on storage additional to those of empty and full reservoir are specified, for instance variable requirements on freeboard and minimum storage level throughout the year as shown in Figure 6*a*.

The procedure is as follows:

1. A straight line connecting the end points A_0 and A_N is computed.

2. The corridor boundaries are checked to see whether any of them are crossed by the line A_0A_N. If no crossing is recorded, the line A_0A_N is the desired shortest path. If X^* crosses A_0A_N, the point of maximum distance of X^* above A_0A_N is identified as a corner point A_i of the shortest path; if X^{**} crosses A_0A_N, the point of maximum distance of X^{**} below A_0A_N is identified as another corner point A_j (Figure 6*b*).

3. The corner point closest to the starting point A_0, in our case A_j, is regarded as an end point of the shortest path in the period $(0, j)$.

4. Steps 1–3 are repeated with A_j replacing A_N.

5. If no additional corner points are identified in the interval $(0, j)$, the straight line A_0A_j is the first segment of the shortest path and the search moves to the next interval (j, i) with A_j and A_i representing the starting and the end points, respectively. If, however, an additional corner point, say A_k (Figure 6*c*), is identified in the period $(0, j)$, the search moves to the interval $(0, k)$.

6. In general, the search always moves forward in time only after the shortest path in the whole past period has been found as shown in Figure 6*b*–6*f*.

The order in which the corner points in Figure 6 were identified is shown in Table 1. A listing of a Fortran-IV program for this procedure is available from the author.

Acknowledgments. I am greatly indebted to S. N. Kritskiy, Institut Vodnykh Problem, Moscow, for information on the origin of the stretched-thread method and for reprints of some of his and M. F. Menkel's old papers. My thanks are also directed to R. J. Cornish, Professor Emeritus, University of Manchester, for his lecture notes from the 1930's on mass-curve analysis, and to J. Laszewski, Universidad del Zulia, Maracaibo, for results of his own work on streamflow mass curves and for references on Schoklitsch's early work. Finally, I would like to express my thanks and appreciation to S.

Kratochvil, Professor Emeritus, Technical University, Brno, for his help with the acquisition of reprints of papers of Czech and Austrian engineers of the beginning of the century. In this rather difficult task I was also helped by two other Czechoslovakian scientists who prefer to remain anonymous and to whom my sincere thanks are due. Stimulating discussions with Myron B. Fiering, Harvard University, Cambridge, Massachusetts, as well as his comments on an earlier draft of this paper contributed to the attempted clarification of some inconsistencies in the treatment of storage mass-curve analysis in recent literature. However, the views expressed herein are my own and need not necessarily coincide in all respects with his perception of the situation.

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(Received September 23, 1977;
revised August 31, 1978;
accepted September 11, 1978.)