ESM 121

Water Science
and Management

Exercise 5:

Optimization and Linear Programing

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Table of Contents

Objective

The objective of this exercise is to provide a set of problems using simple optimization techniques as well as the basics of linear programming using a linear optimization solver in Excel.

Useful Formulas

Find the decision variables, ***x***, that optimizes (maximizes or minimizes) an objective function.

For instance, to minimize:

$$\underset{\mathbf{x}}{\text{minimize}} \quad f(\mathbf{x})$$

← **Objective function**

← **Decision variables**

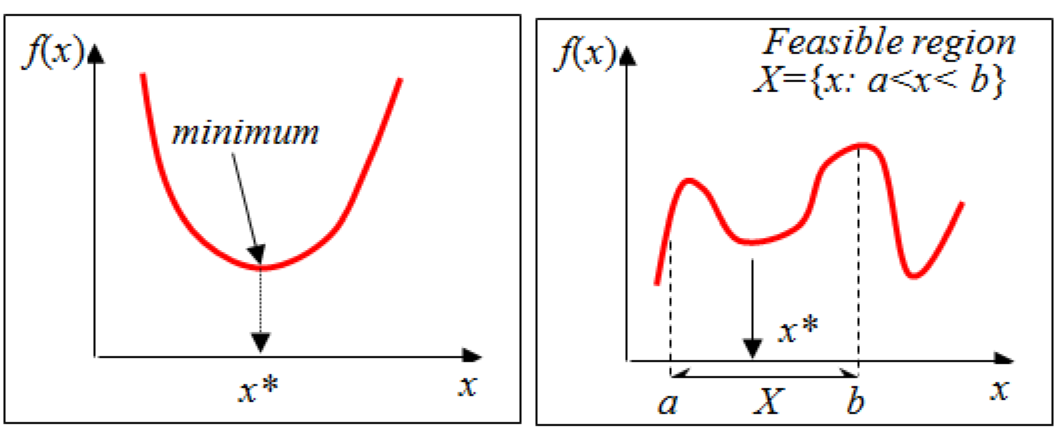
subject to

$$\mathbf{x} \in X$$

$$Ax + b \leq c$$

← **Constraint set**

$$x \geq 0$$



Steps for Linear Programming

In order to solve linear programing, these are 7 steps that you should follow to identify the maximum or minimum value for the objective function at hand:

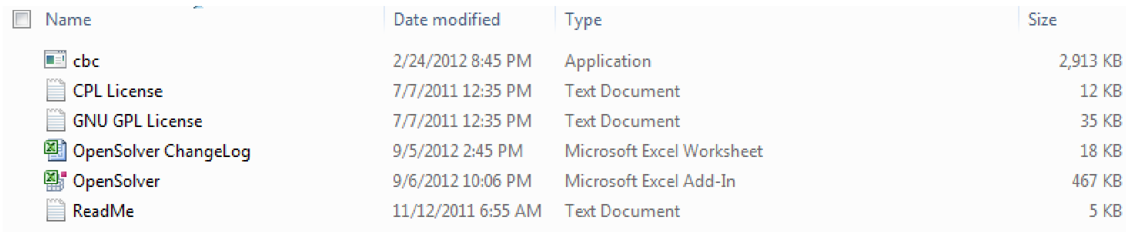
1. Define the *optimization purpose*. Is the objective to **Maximize or Minimize**?
2. Define *Objective Function*, or in other words, write the Objective Function into an equation. If the equation is linear ($Z=ax_1+bx_2$), it can be solved through a linear programming. If not, other techniques of non-linear programing can be used to solve this type of problem.
3. Define the Constraints. Write the constraints into inequalities, so they can be used to define the Feasible Region. Notice that the statement "the value of x is greater than (or at least) 20" means $x\geq20$, and the statement "the value of x is smaller than (or less than) 20" means $x\leq20$
4. Define the Feasible Region. Use the constraints (inequalities) to bound the feasible region. For constraints with the form "$ax_1+bx_2\leq C$", convert them into equations by dropping the "≤" or "≥" symbol and adding the equal sign "=", such as "$ax_1+bx_2=C$". Then solve the equation for $x_1=0$ to obtain the point where the linear equation crosses the x_1 axis. Similarly, solve for $x_2=0$ to obtain the point where the linear equation cross the x_2 axis.
5. Obtain the vertices of the feasible region. Do this by identifying the points in the feasible region and, for the constraints with the form "ax1+bx2≤C", convert them into equations "ax1+bx2=C" and obtain the value of the unknown variable, either x1 or x2, by solving the equation.
6. Substitute vertices into the Objective Function. Substitute the values of the variables (x_1 and x_2) at the vertices into the objective function equation.
7. Select the values of the variables that are the maximum or minimum for the objective function, depending on the definition of the objective function (Step 1).

Installing Open Solver - Model

Installing the Open Solver in excel.

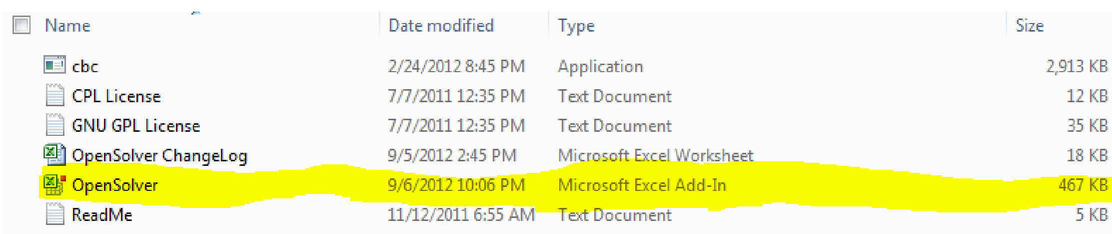
1.- Copy the OpenSolver21.zip into your computer.

2.- Unzip this file



Name	Date modified	Type	Size
cbc	2/24/2012 8:45 PM	Application	2,913 KB
CPL License	7/7/2011 12:35 PM	Text Document	12 KB
GNU GPL License	7/7/2011 12:35 PM	Text Document	35 KB
OpenSolver ChangeLog	9/5/2012 2:45 PM	Microsoft Excel Worksheet	18 KB
OpenSolver	9/6/2012 10:06 PM	Microsoft Excel Add-In	467 KB
ReadMe	11/12/2011 6:55 AM	Text Document	5 KB

3.- Double click on the file: "OpenSolver" which is the add-In



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cbc	2/24/2012 8:45 PM	Application	2,913 KB
CPL License	7/7/2011 12:35 PM	Text Document	12 KB
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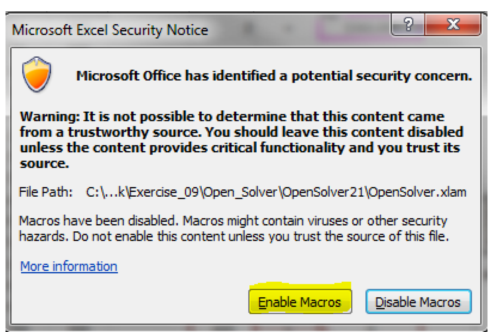
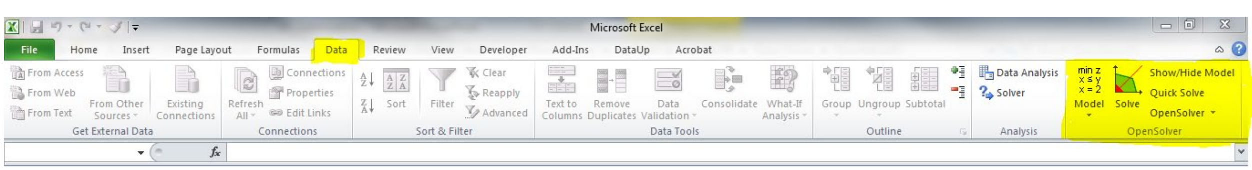
OpenSolver ChangeLog	9/5/2012 2:45 PM	Microsoft Excel Worksheet	18 KB
OpenSolver	9/6/2012 10:06 PM	Microsoft Excel Add-In	467 KB
ReadMe	11/12/2011 6:55 AM	Text Document	5 KB

4.- If a dialogue-window pops up, please select "enable macros"



5.- On the data menu of Excel you should see the Open Solver add-in in the right corner of the menu.



6.- Finally Open the file Ex_5.xlsx

Exercise 1 (Adapted from McKinney).

Based on an annual water allocation of 1,800 acre-feet (AF), an irrigation district wants to know how they can maximize their profits by growing two types of crops, Crop A and Crop B. The profits can be obtained by summing the profit (C_A and C_B) times the acreage (X_A and X_B) for each crop. The irrigation district has constraints. The first constraint is the water use, which is the sum of the water requirement times the acreage for each crop, which must be equal or less than the water allocation (1,800 AF). Second, there are limits to growing one crop or another. The maximum acreage on which to grow crop *A* is 400 acre, and crop *B* is 600 acres. Lastly, the acreage must be a positive number for both crops. Table 1 shows the water requirement, expected profit per acre and maximum area cultivated for each crop.

Table 1

	Crop A	Crop B
Water requirement (Acre feet/acre)	3	2
Profit ($/acre)	300	500
Max area (acres)	400	600

To be turned in:

1) Write down the objective equations and constraints for this problem. Take a look at the "Optimization Presentation", this exercise was explained in this presentation.
2) Copy the chart for the feasible region located in Ex_5.xlsx file tab "1 Maximize"
3) What was the solution (Maximum value for the objective function *Z*) in the presentation?

Now, let's work on the Ex_5.xlsx file. Open the tab "1 Maximize". The right side of that tab should look like Figure 1. Every cell that is orange means you must declare some information there.

Variables and Objective Function

First, let's start with the *Variables*. We need to give the model a first guess of the variable, usually any value inside the feasible region and the model will take care to later give us the right result once we have run it. For X_A lets define an initial value of 400 (cell T4=400), and for X_B a value of 600 (cell T5=600), see Figure 2.

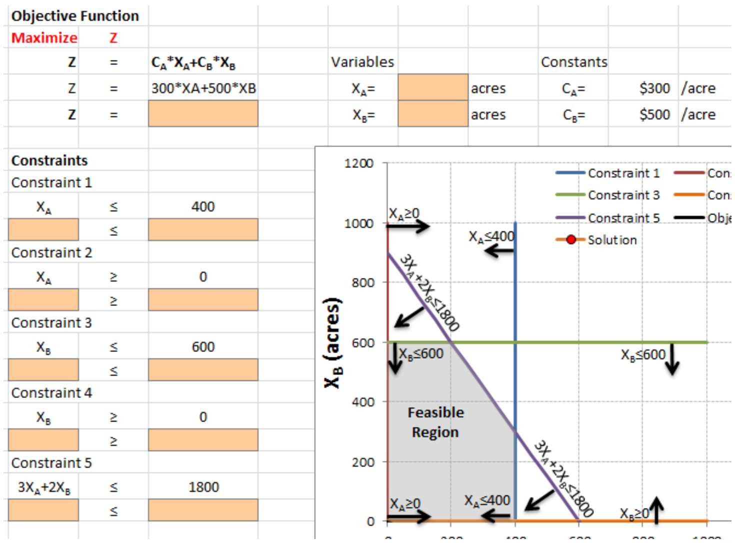


Figure 1

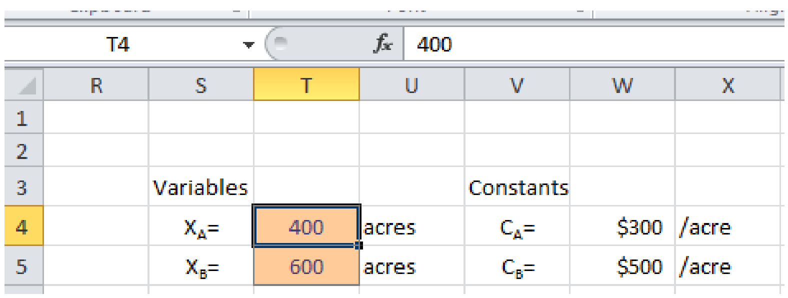


Figure 2

Now let's define the Objective function in cell Q5 as the sum of the multiplication of the acreage times the profits for each crop (cell Q5 =T4*W4+T5*W5), see Figure 3.

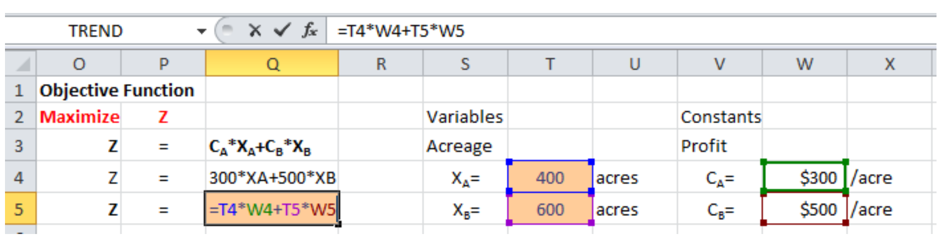


Figure 3

Constraints

Now let's work on the *Constraints*. Constraint 1 specifies that the acreage of crop a (X_A) must be equal to or less than 400 acres. Link cell O10 with the value of *XA* (cell T4) by typing in cell O10 "=T4"; see Figure 4. Type in cell Q10 the constraint value of 400 (Figure 5).

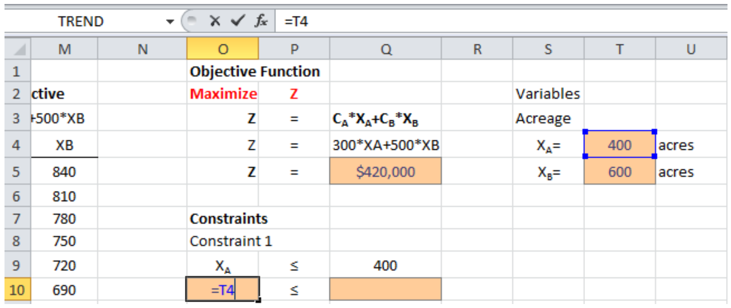


Figure 4

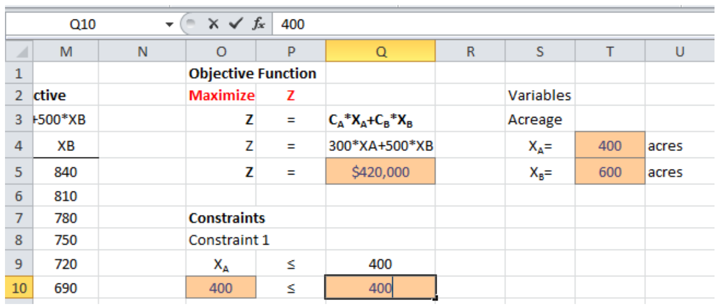


Figure 5

Constraint 2 specifies that the acreage of crop A (XA) must be equal to or greater than 0 acres. Link cell O13 with the value of XA (cell T4) by typing in cell O13 "=T4"; see Figure 6. Type in cell Q13 the constraint value of 0 (Figure 7).

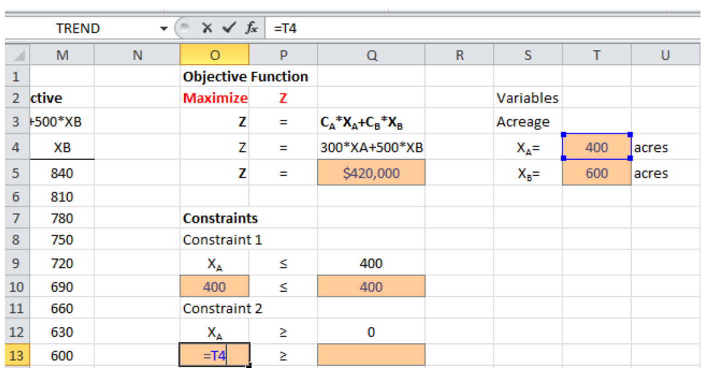


Figure 6

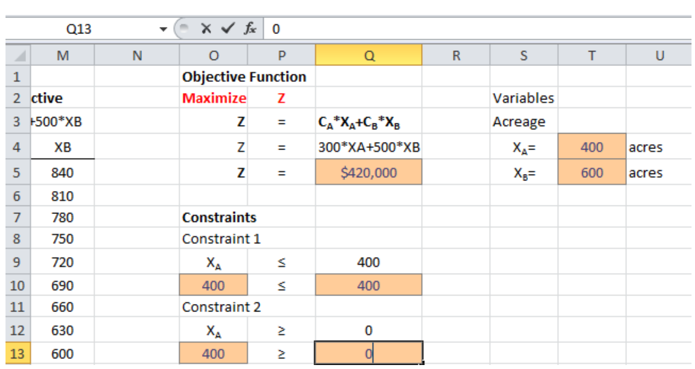
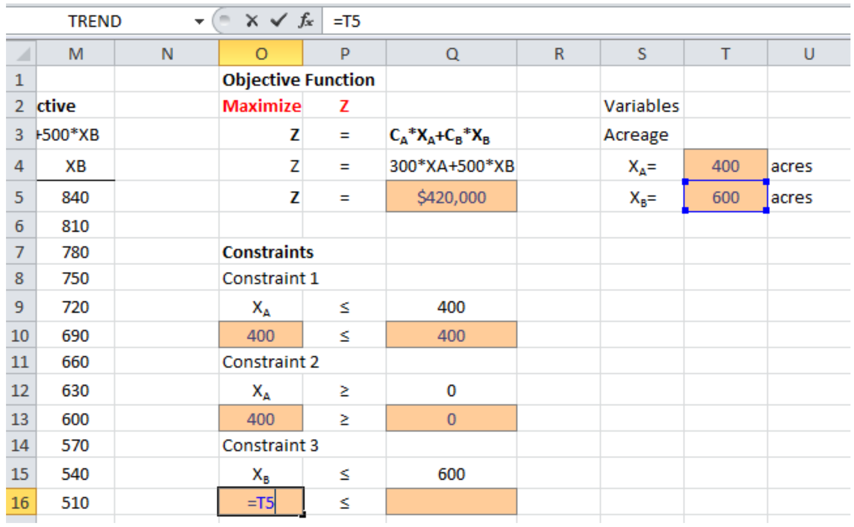


Figure 7

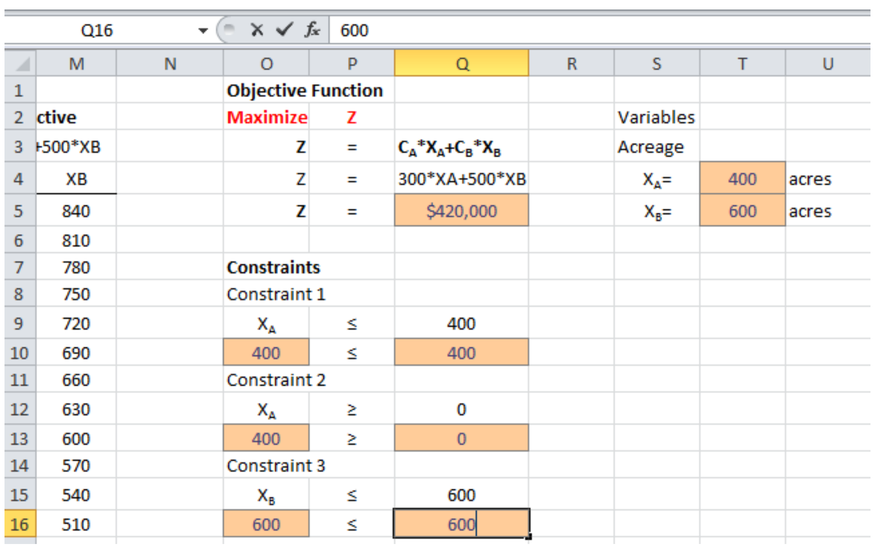
Constraint 3 specifies that the acreage of crop B (X_B) must be equal to or less than 600 acres. Link cell O16 with the value of X_B (cell T5) by typing in cell O16 "=T5"; see Figure 8. Type in cell Q16 the constraint value of 600 (Figure 9).



TREND | =T5

	M	N	O	P	Q	R	S	T	U
1			**Objective Function**						
2	**ctive**		**Maximize**	**Z**			Variables		
3	+500*XB		**Z**	=	C_A*X_A+C_B*X_B		Acreage		
4	XB		Z	=	300*XA+500*XB		X_A=	400	acres
5	840		**Z**	=	$420,000		X_B=	600	acres
6	810								
7	780		**Constraints**						
8	750		Constraint 1						
9	720		X_A	≤	400				
10	690		400	≤	400				
11	660		Constraint 2						
12	630		X_A	≥	0				
13	600		400	≥	0				
14	570		Constraint 3						
15	540		X_B	≤	600				
16	510		=T5	≤					

Figure 8



Q16 | 600

	M	N	O	P	Q	R	S	T	U
1			**Objective Function**						
2	**ctive**		**Maximize**	**Z**			Variables		
3	+500*XB		**Z**	=	C_A*X_A+C_B*X_B		Acreage		
4	XB		Z	=	300*XA+500*XB		X_A=	400	acres
5	840		**Z**	=	$420,000		X_B=	600	acres
6	810								
7	780		**Constraints**						
8	750		Constraint 1						
9	720		X_A	≤	400				
10	690		400	≤	400				
11	660		Constraint 2						
12	630		X_A	≥	0				
13	600		400	≥	0				
14	570		Constraint 3						
15	540		X_B	≤	600				
16	510		600	≤	600				

Figure 9

Constraint 4 specifies that the acreage of crop B (X_B) must be equal or greater than 0 acres. Link cell O19 with the value of X_B (cell T5) by typing in cell O19 "=T5"; see Figure 10. Type in cell Q19 the constraint value of 0 (Figure 11).

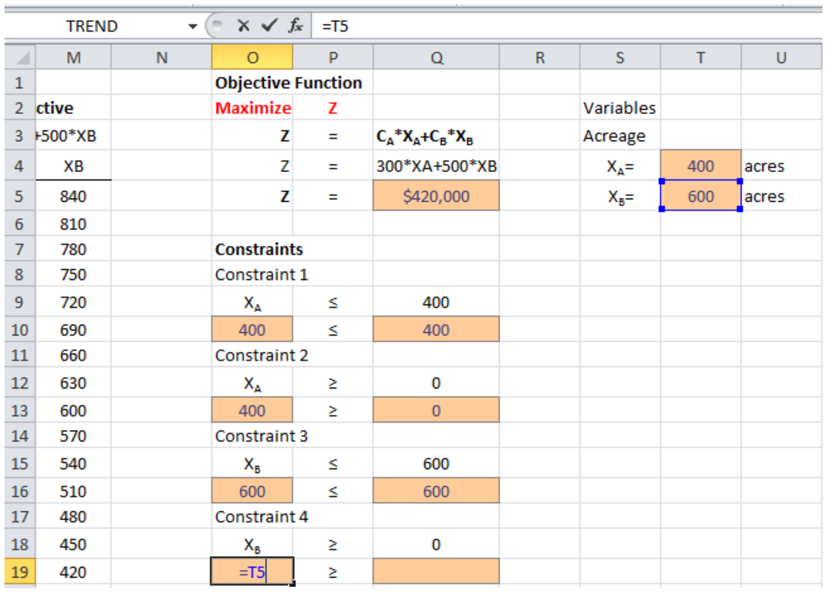


Figure 10

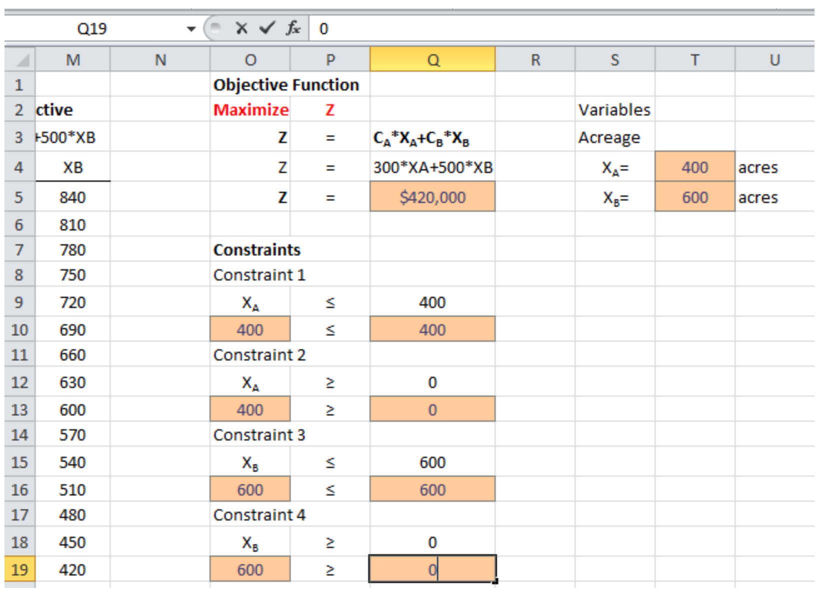


Figure 11

Constraint 5 specifies that the sum of the water use in crop A ($3*X_A$) and crop B ($2*X_B$) must be equal or less than 1800 acre-feet. In cell O22 we have to write this equation as follow: "=Z4*T4+Z5*T5"; see Figure 12. Type in cell Q19 the constraint value of 1800 (Figure 13).

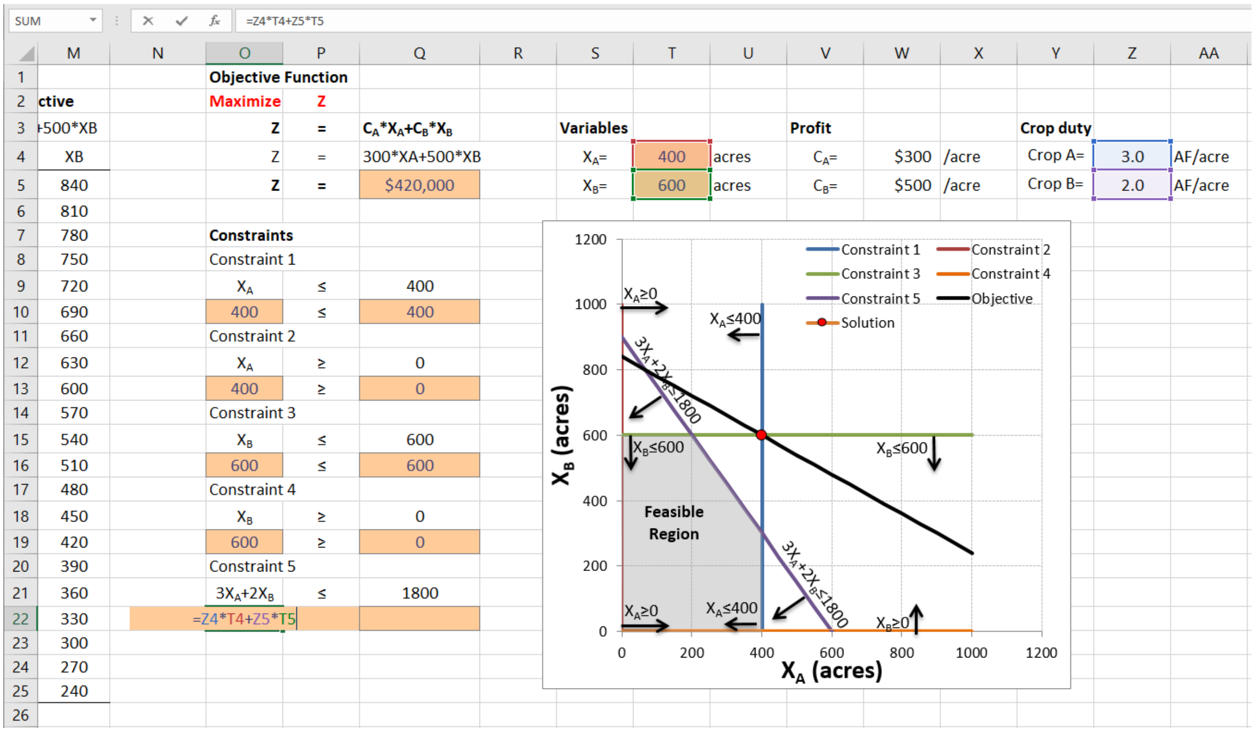


Figure 12

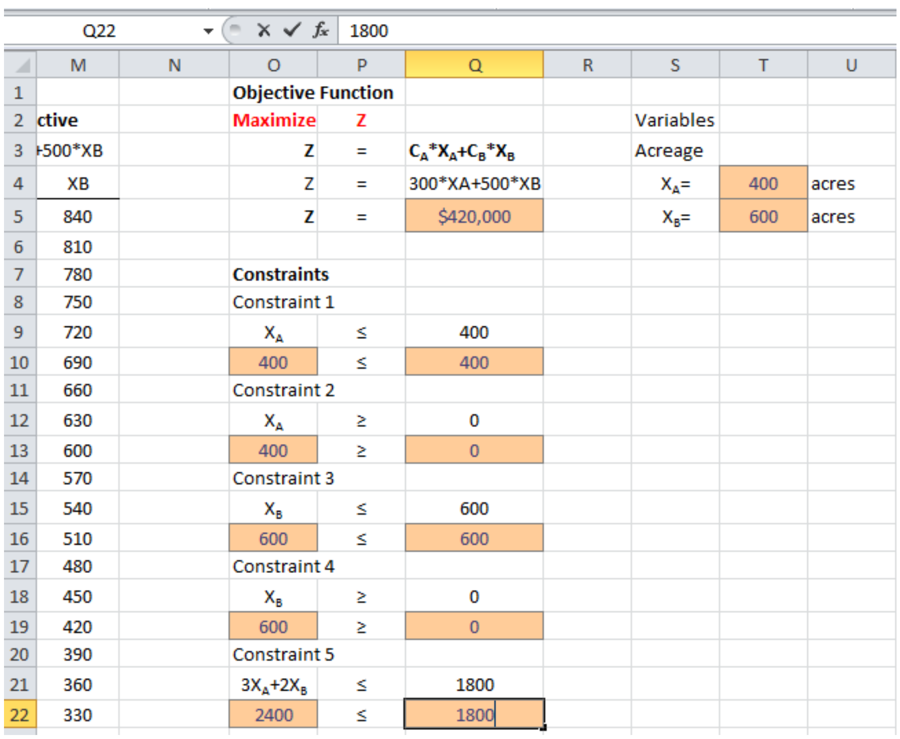


Figure 13

Save your spreadsheet (Ctrl+S).

Defining the Optimization Linear Program

Now, let's define the linear program model. Go to the Data menu and click on "Min Z x≤y x=2 Model" icon (Figure 14)

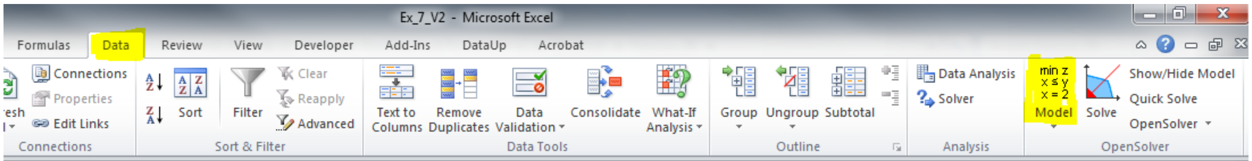


Figure 14

A dialogue window will appear, displaying the "Open Solver - Model" (Figure 15).

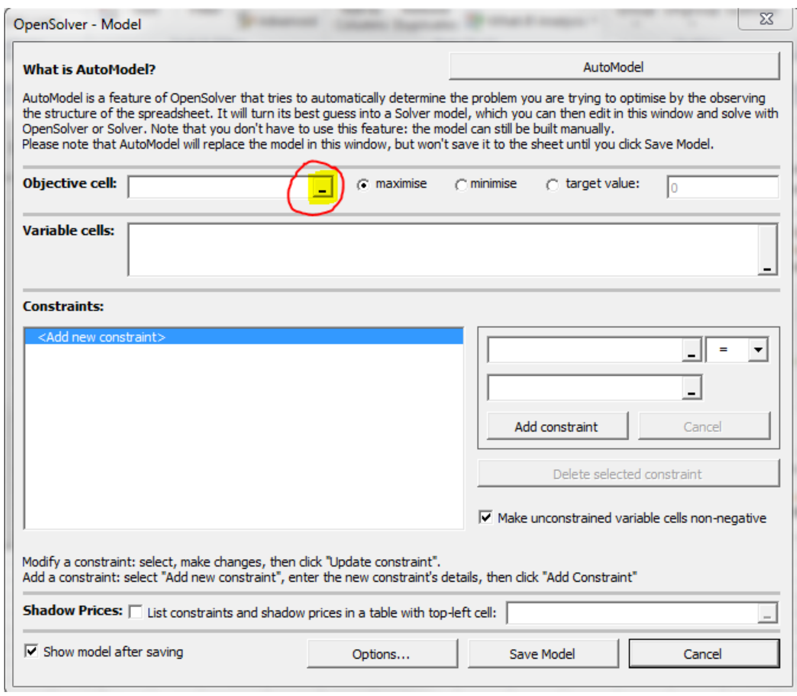


Figure 15

Go to the Objective cell section, and click on the icon to browse the objective cell (Figure 15). Select the cell Q5, which is the cell with the equation of the objective function (Figure 16).

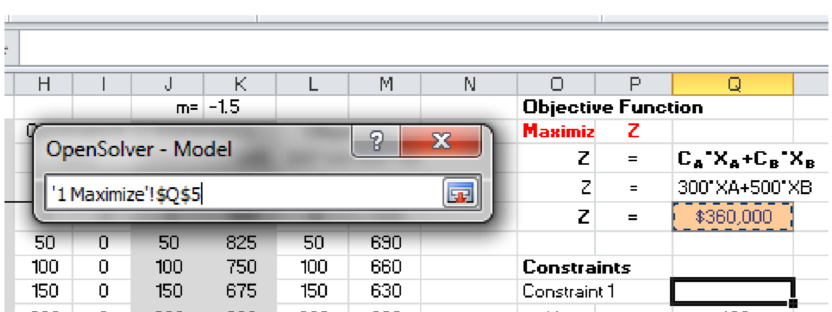


Figure 16

When you come back to the dialogue window, make sure to select "maximize" (Figure 17).

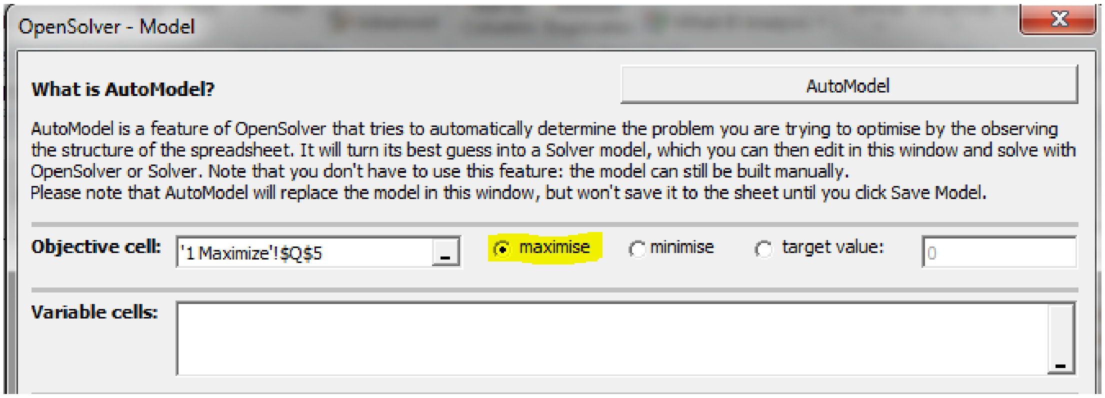


Figure 17

Now click on the icon to choose the cells that will be the variables (Figure 18). Select cells T4 and T5 (Figure 19).

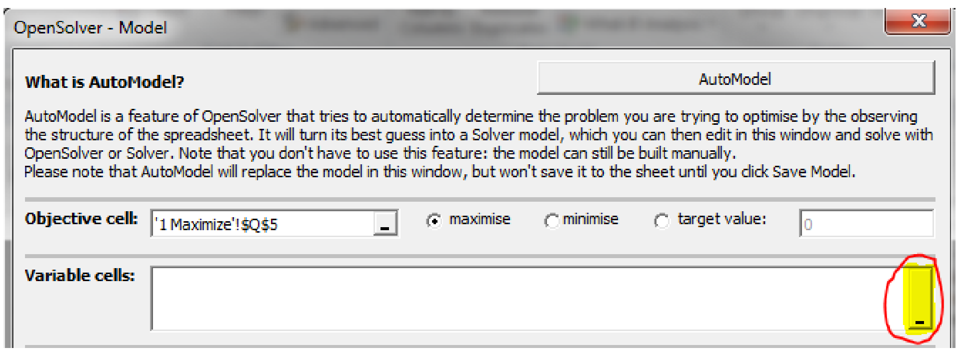


Figure 18

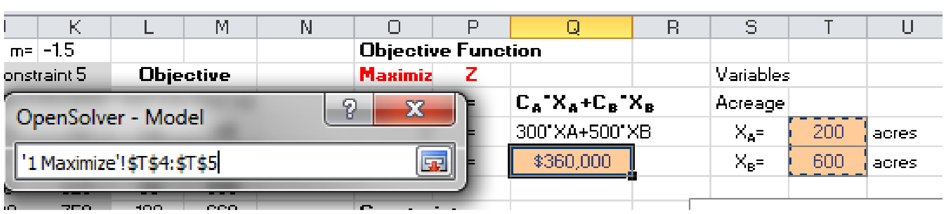


Figure 19

Now, we will add the constraints to the model. Let's start with the first constraint ($X_A \leq 400$), go to the constraints section and click on the icon to browse for the left hand side of the constraint (Figure 20). Select the cell O10 (Figure 21), then go back to the dialogue menu. Select the symbol of equal or less than "≤" (Figure 22). Now click on the icon to select the right side of the inequality (Figure 23), cell Q10 (Figure 24) and go back the dialogue window. Click on "Add Constraint" (Figure 25). The inequality Cell O10 ≤ Cell Q10 should have appeared in the left side of the Constraints section of the dialogue window (Figure 26). We just have finished declaring the first constraint.

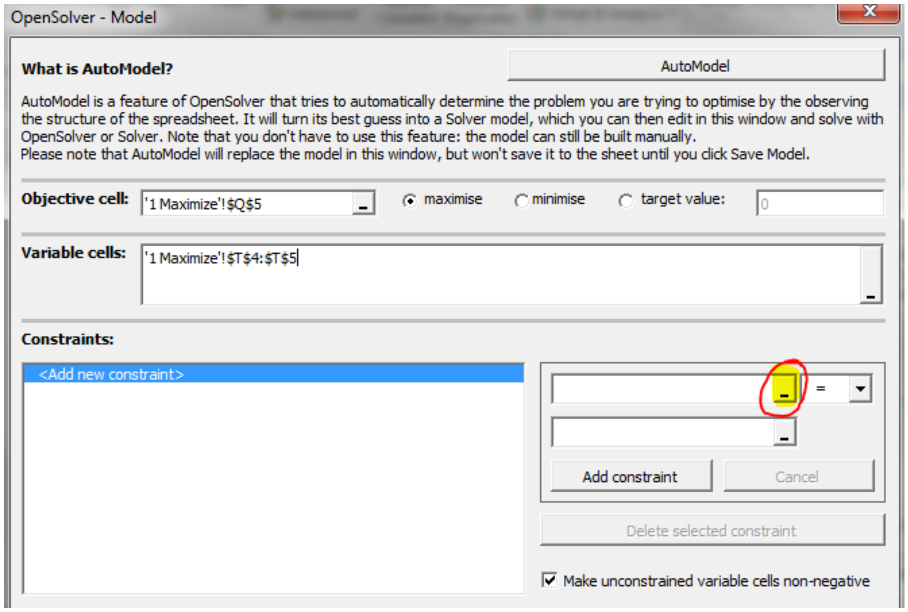


Figure 20

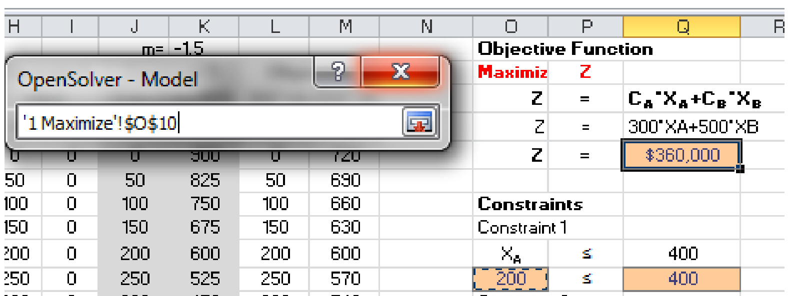


Figure 21

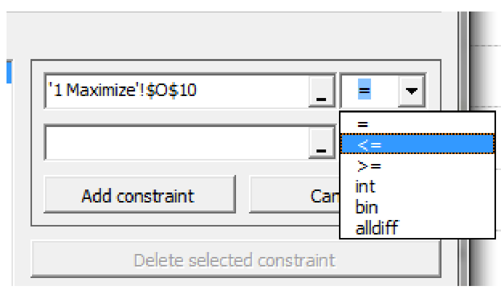


Figure 22

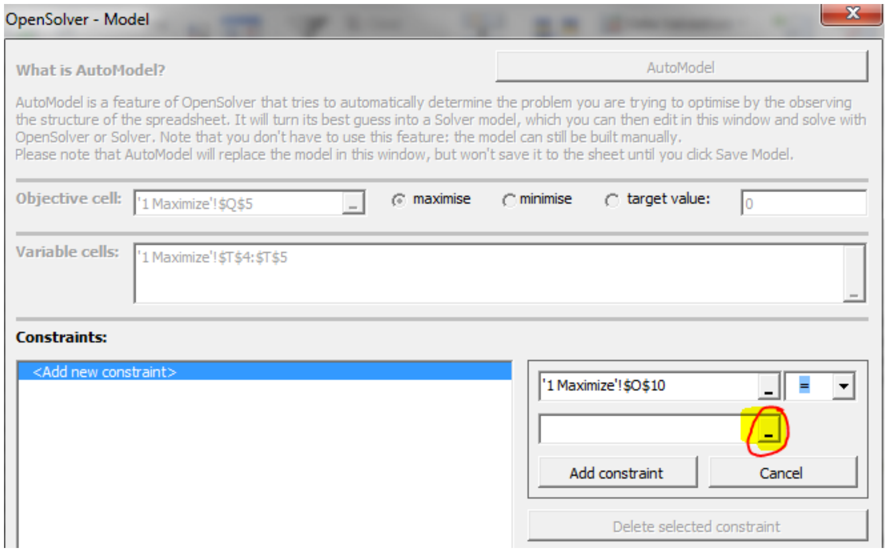


Figure 23

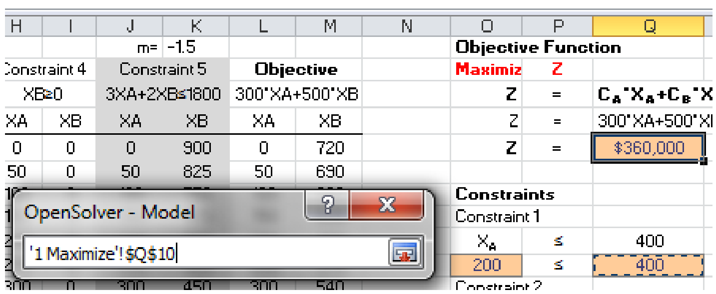


Figure 24

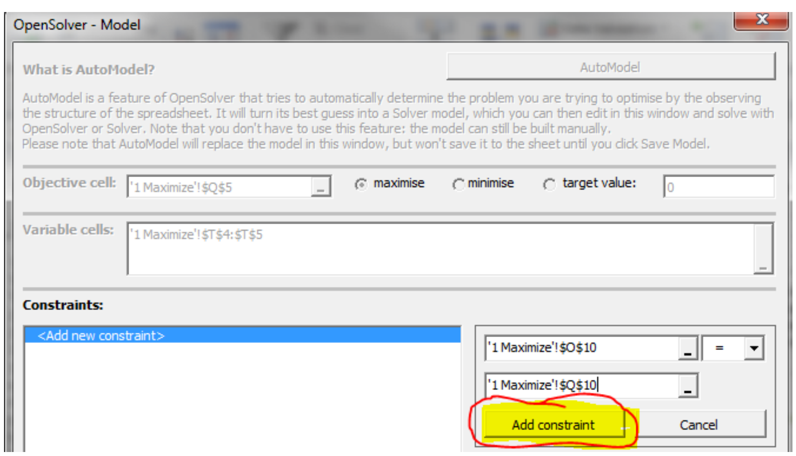


Figure 25

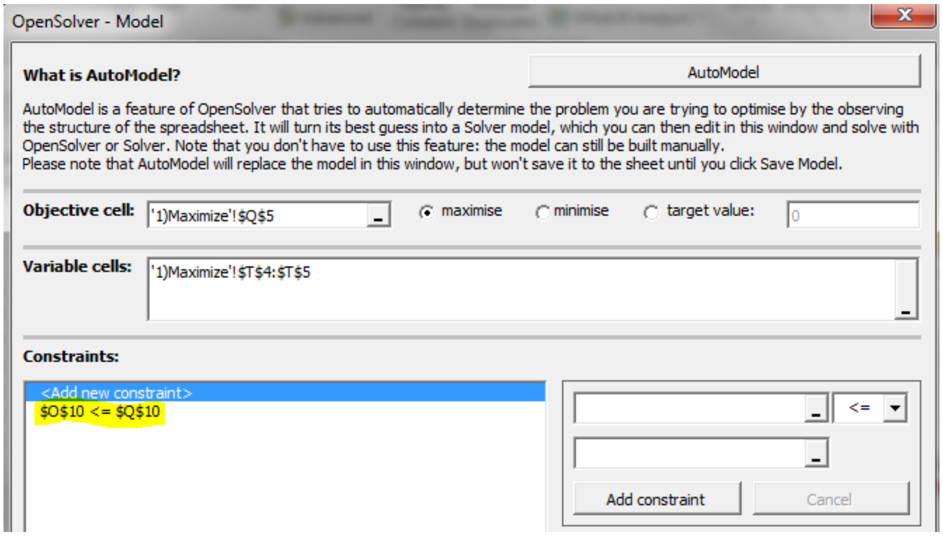


Figure 26

To declare the second constraint ($X_A \geq 0$), click on the icon to browse for the left hand side of the constraint (Figure 20). Select the cell O13 (Figure 27), then go back to the dialogue menu. Select the symbol of equal or greater than "≥" (Figure 22). Now click on the icon to select right-side part of the inequality (Figure 23), select cell Q13 (Figure 29) and go back the dialogue window. Click on "Add Constraint". The inequality Cell O13 ≥ Cell Q13 should have appeared in the left side of the Constraints section of the dialogue window (Figure 30).

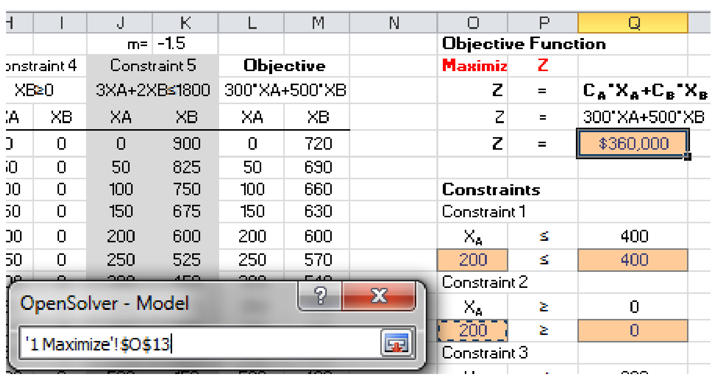


Figure 27

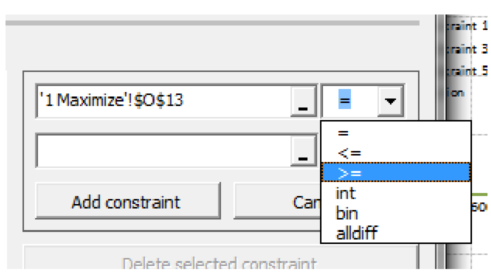


Figure 28

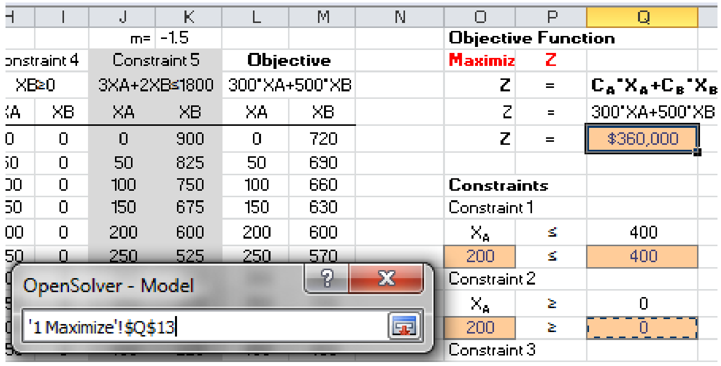


Figure 29

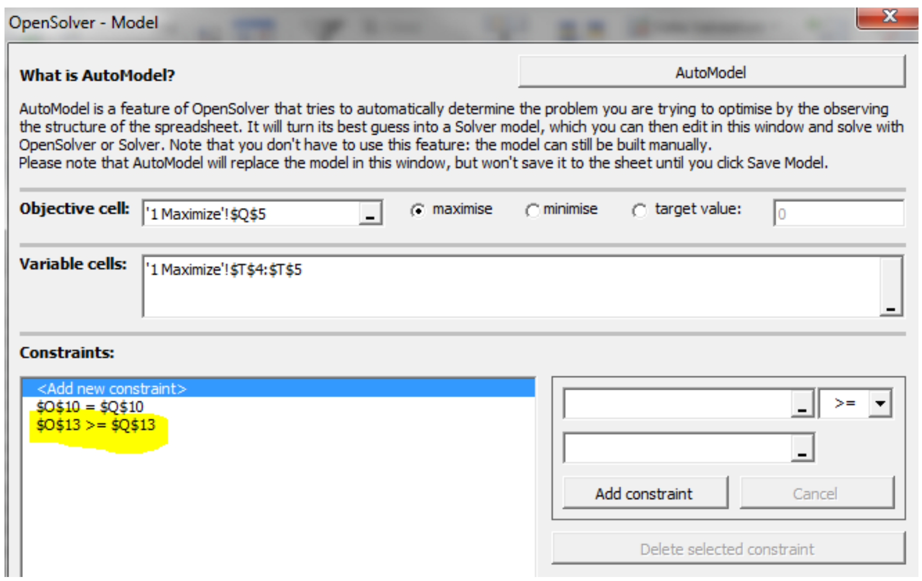


Figure 30

To declare the third constraint ($X_B \leq 600$), click on the icon to browse for the left hand side of the constraint (Figure 20). Select the cell O16 (Figure 31), then go back to the dialogue menu. Select the symbol of equal or greater than "≤" (Figure 32). Now click on the icon to select right-side part of the inequality (Figure 23), select cell Q16 (Figure 33) and go back the dialogue window. Click on "Add Constraint". The inequality of cell O16 ≤ Cell Q16 should have appeared in the left side of the Constraints section of the dialogue window (Figure 34).

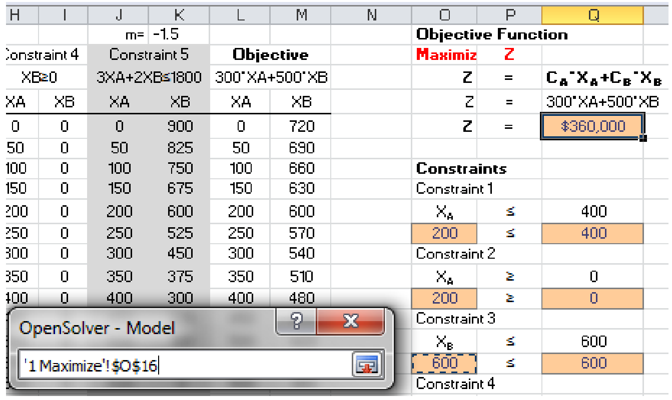


Figure 31

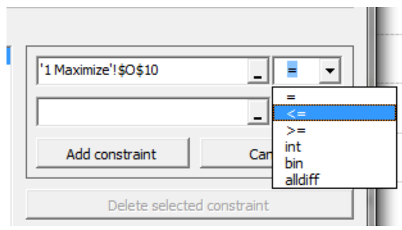


Figure 32

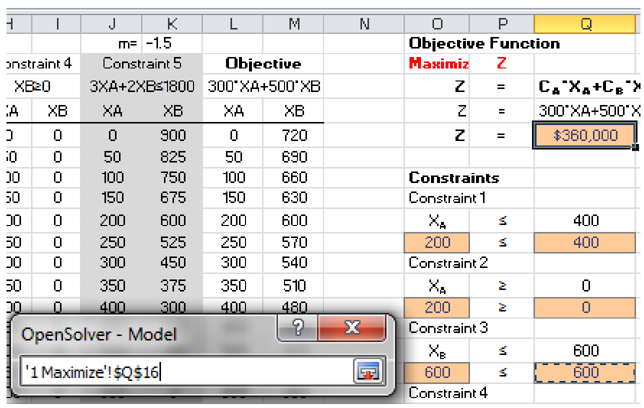


Figure 33

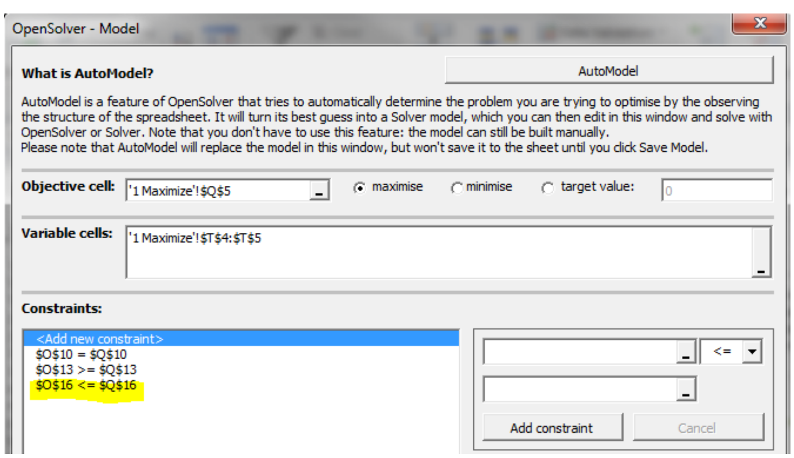


Figure 34

To declare the fourth constraint ($X_B \geq 0$), click on the icon to browse for the left hand side of the constraint (Figure 20). Select the cell O19 (Figure 35), then go back to the dialogue menu. Select the symbol of equal or greater than "≥" (Figure 36). Now click on the icon to select right-side part of the inequality (Figure 23), select cell Q19 (Figure 37) and go back the dialogue window. Click on "Add Constraint". The inequality Cell O19 ≥ Cell Q19 should have appeared in the left side of the Constraints section of the dialogue window (Figure 38).

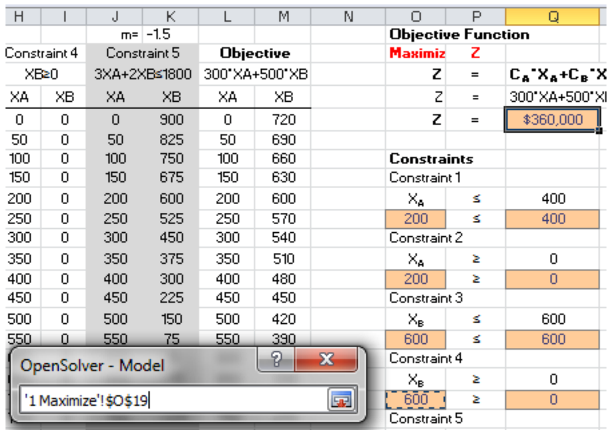


Figure 35

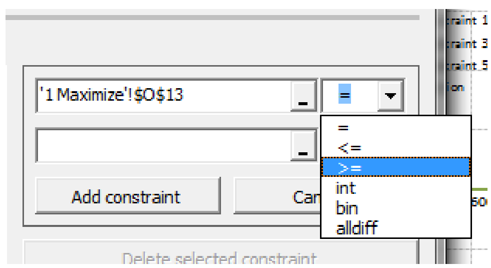


Figure 36

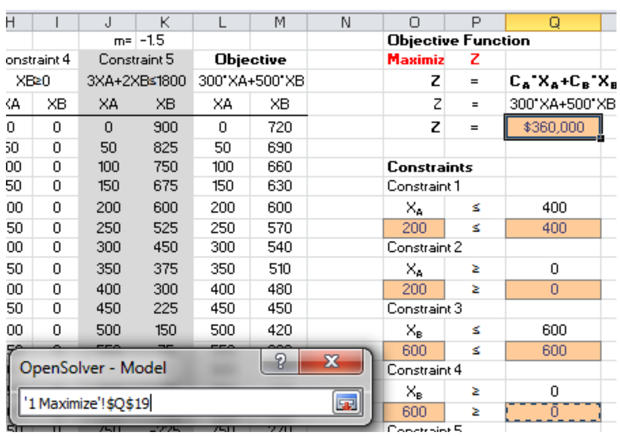


Figure 37

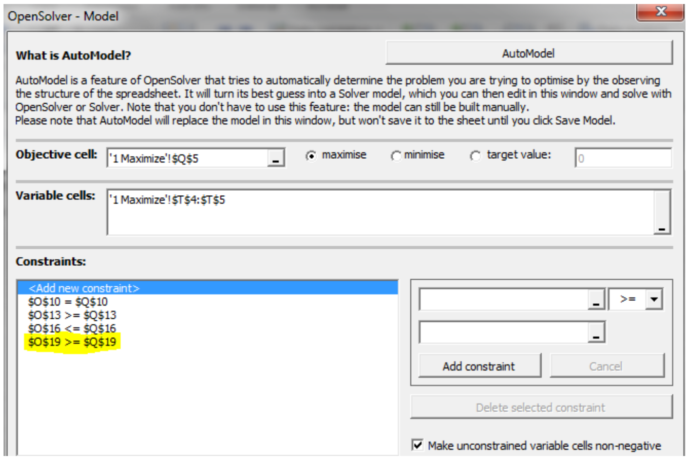


Figure 38

To declare the fifth constraint ($3X_A+2X_B \leq 1800$), click on the icon to browse for the left hand side of the constraint (Figure 20). Select the cell O22 (Figure 39), then go back to the dialogue menu. Select the symbol of equal or greater than "≤" (Figure 40). Now click on the icon to select right-side part of the inequality (Figure 23), select cell Q22 (Figure 41) and go back the dialogue window. Click on "Add Constraint". The inequality of cell O22 ≥ Cell Q22 should have appeared in the left side of the Constraints section of the dialogue window (Figure 42).

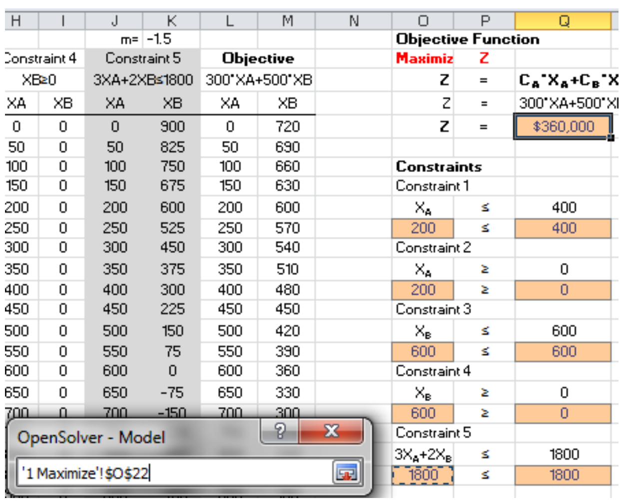


Figure 39

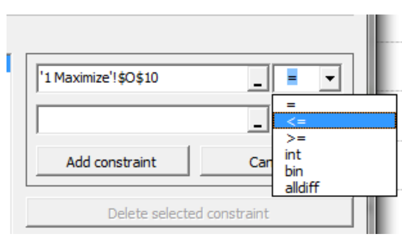


Figure 40

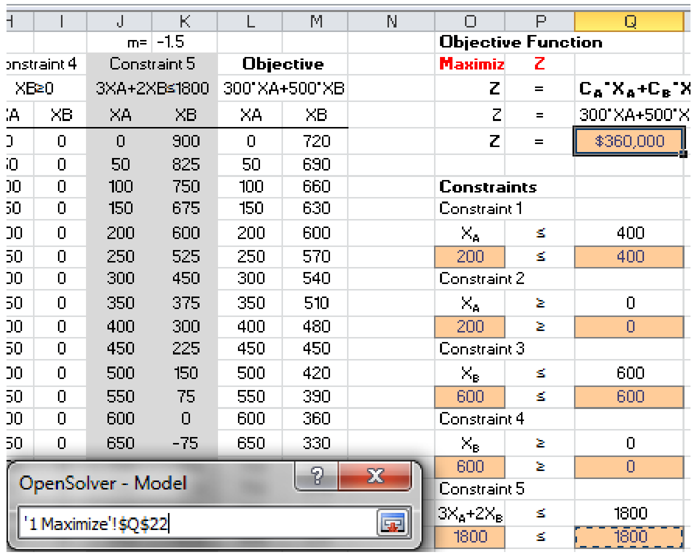


Figure 41

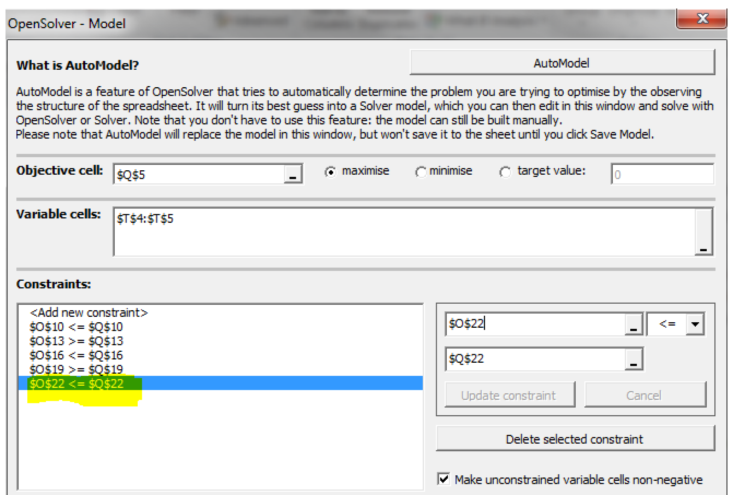


Figure 42

Check the box that says “List constraints and shadow prices in a table with top left cell:” and select the cell “AC8” (Figure 43)

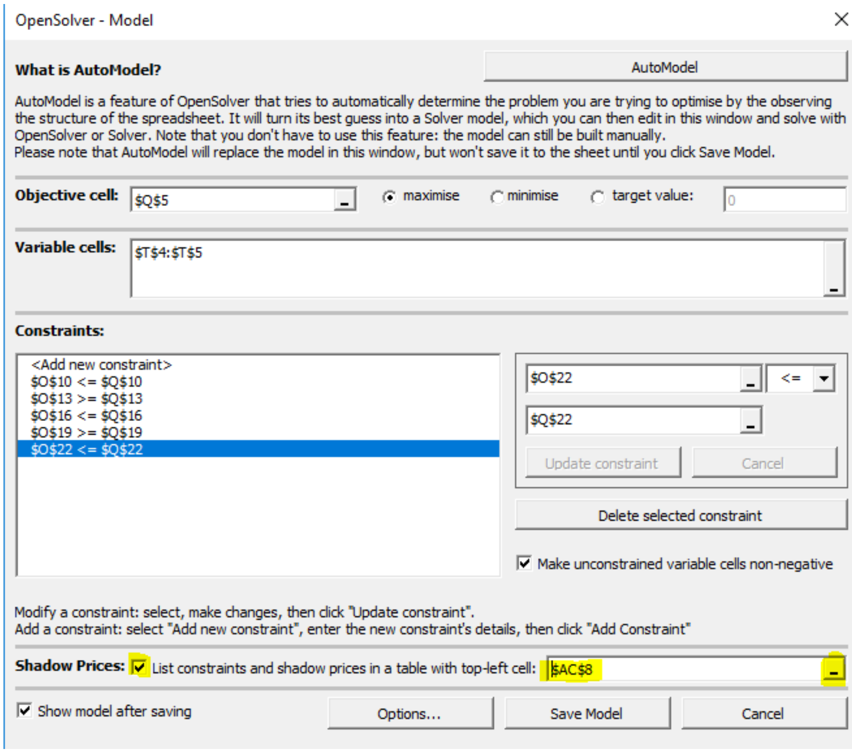


Figure 43

Click on “Save Model” (Figure 44). A Series of colorful rectangles should have appeared in your Spreadsheet (Figure 45). Notice that the Objective Function (cell Q5) has a “Max” label on top of it. Also, notice that the variables have a pink rectangle on top. In addition, for each constraint the right and left side of the equations/inequality are related with a line and a symbol (“≥”, “≤” or “=”), Double check that these symbols are correct.

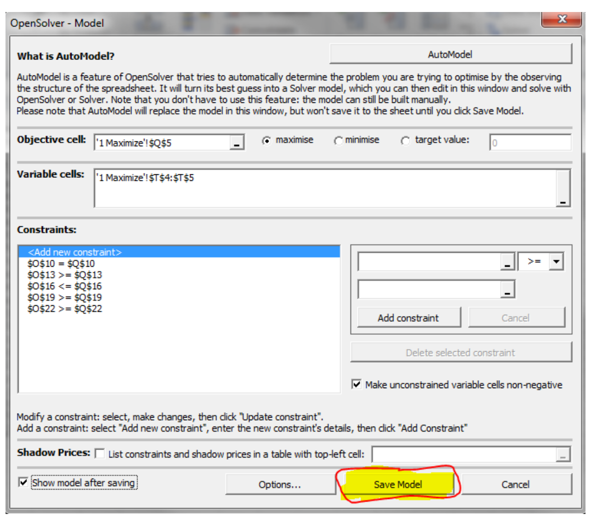


Figure 44

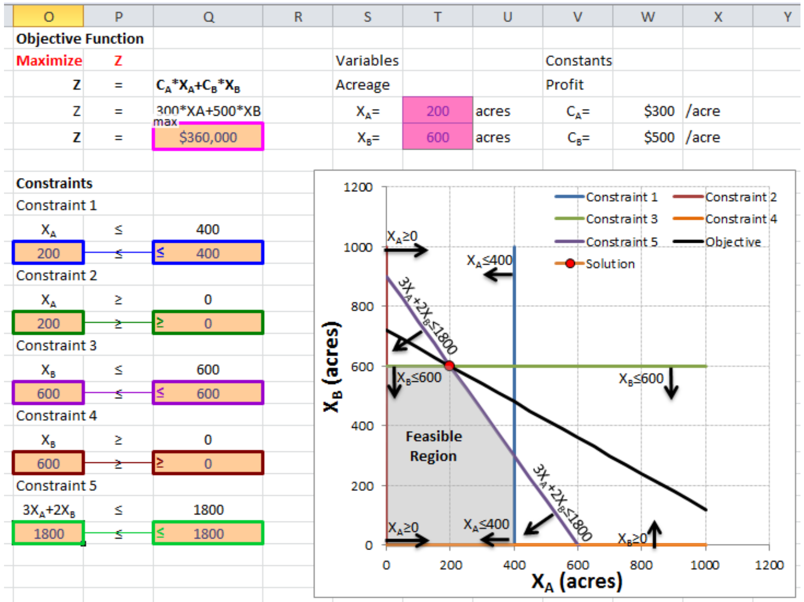


Figure 45

Running the Linear Programming Model

It is time to run the model!!! Click on the Solve icon (Figure 46) of the Open Solver. After the screen blinks you will notice that numbers have changed. The maximum profit that can be obtained are **$360,000 (Cell Q5)**!!! The values for the variable have changed, X_A=200 and X_B=600, as we calculated in the presentation at class!!!

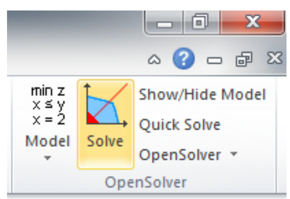


Figure 46

To be turned in:

1) Take a look at the shadow values. For the constraint in cells O16<=Q16 (X_B≤600) the shadow value is equal to -300, what does it mean? *Do the following to get a hint for the answer of this question*: Change the value of cell Q16 to 601 and run the model. What is the new value of the objective function (cell Q5)? Now change it to 602, and take a look at the objective function value. How much the objective value changed for 601? And for 602? And for 603? Does the shadow price tell you how much the objective function will change by a unit increase in the constraint?
2) A screenshot of the model, like Figure 45.
3) Now let's invert the values so the Profits for C_A (cell W4) are equal to $500/acre and for C_B (cell W5) are equal to $300/acre. Click on Solve. What are the results for X_A and X_B? Does the value of the profits change the results for the optimal solution? If so, do you think changing the market prices can change the optimal solution of linear systems?
4) A screenshot of the new model with the profit values changed.

Exercise 2 (Adapted from Loucks and van Beek)

A city planner wants to know the minimum cost of waste removal (WR) at Sites 1 and 2 upstream of a recreational park (Figure 47). The cost of waste removal can be obtained by summing the cost of 100% (C_1 and C_2) times the fraction of removal for each site (X_1 and X_2). According to a pre-design, the fraction of WR at site 1 (X_1) must be equal to or greater than 0.8 but equal to or less than 1. The fraction of WR at site 2 (X_2) must be equal to or less than 1. There is another constraint expressed as the equation: $X_1 + 1.3\,X_2 \geq 1.8$.

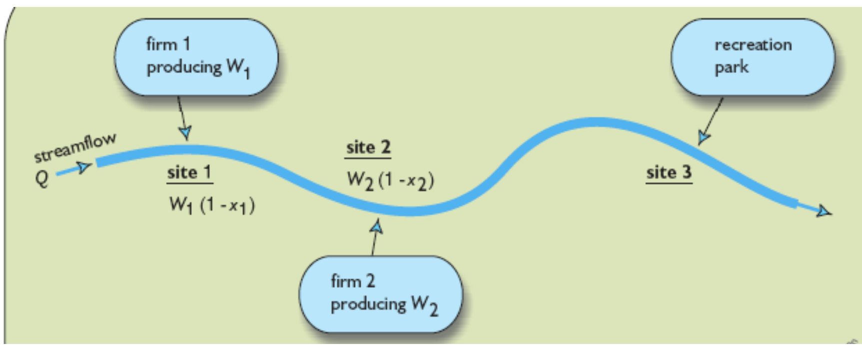


Figure 47.

Use the "Optimization" presentation in class to solve this problem. The optimization model is as follows:

$$\text{Minimize } Z$$
$$Z = C_1(x_1) + C_2(x_2)$$
$$\textit{Where}: C_1 = 200; C_2 = 100$$
$$\text{Subject to}$$
$$x_1 \geq 0.8$$
$$x_1 \leq 1$$
$$x_2 \geq 0$$
$$x_2 \leq 1$$
$$x_1 + 1.3x_2 \geq 1.8$$

Using the Excel spreadsheet Ex_5.xlsx in Tab "2) Minimize", create a linear program using the Open Solver Model. Don't forget to select "**Minimize**" instead of "Maximize" (Figure 17) when creating the linear model.

To be turned in:

1) A screenshot of your linear model, such as Figure 44.
2) What is the minimum cost of removal (the optimum value of the objective function) obtained once the model has been run? How do your results compare with the ones in the presentation (Slide 20, minimum cost of removal equal to $238K)

Exercise 3 The perfect outfit!

Summer is coming and you are looking for the perfect outfit to wear, while, at the trying not to over spend. You have a total budget of $130. This is a suggested number of minimum and maximum number of pieces that you want to buy for each article.

Article	Minimun # of pieces	Maximum # of Pieces
Jeans (C_{Jeans})	1	3
T-Shirts ($C_{T\text{-}shirts}$)	1	4

Other constraints are that you want to buy equal or more T-shirts than jeans ($C_{T\text{-shirts}} \geq C_{\text{Jeans}}$), and that the total amount spent should not be more than your dedicated budget $130.

To be turned in:

Following the seven steps of linear programing (page 4 of this exercise)

1) Write down the *objective function* and the constraints
2) Draw the *feasible region*. You can use excel, or a simple piece of paper and scan it, or some engineering paper.
3) Obtain the vertices of the feasible region. Substitute the values of the vertices into the objective function. Submit a table showing the vertices of the feasible region as well as the value of the objective function for each pair of C_{Jeans} and $C_{T\text{-}shirts}$ values (see slides 13 and 19 of the presentation of "Optimization").
4) Based on the previous analysis, what is the combination Jeans and T-shirts that you should buy?

Now, let's use the Open Solver model of excel to solve this problem. Use the file Ex_5.xlsx - tab "3 Maximize" to create a linear optimization model. You can use as initial values for C_{Jeans} (cell G5) and $C_{T\text{-}shirt}$ (cell G6) 1 and 1, respectively. Calculate amount spend in cell D6 (for the objective function) and B23 (for the constraint) by multiplying the cost of the jeans times the number of jeans (J5xG5) plus the cost of T=shirts times the number of T-shirts (J6xG6). When creating the linear optimization model in Open Solver, don't forget to select "Maximize" (Figure 17). In addition, you will have to declare in the optimization model that the number of jeans C_{Jeans} (cell G5) and T-shirts $C_{T\text{-}shirts}$ (cell G6) should be integers (see figure 48).

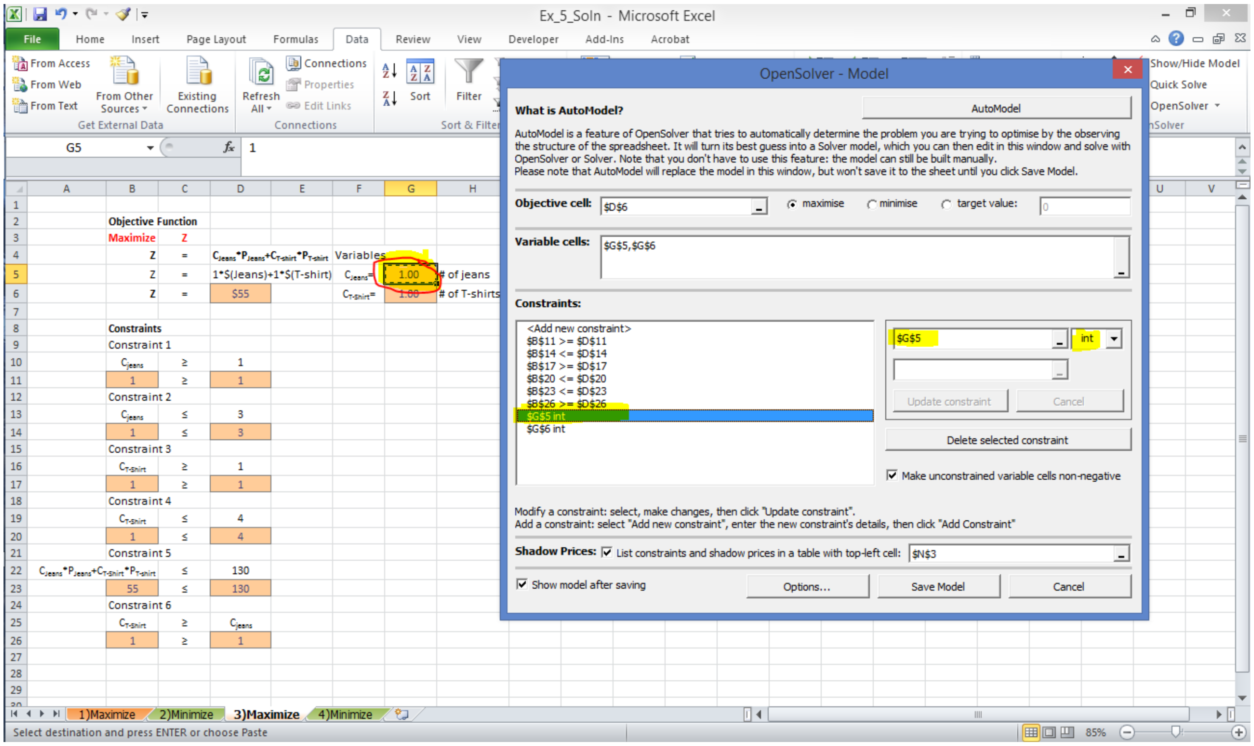


Figure 48

5) Turn in a screenshot of your linear model, similar to Figure 44.
6) How much will you spend (the optimum value of the objective function) obtained once the model has been run (cell D5)?
7) How many Jeans and T-shirts you can buy C_{Jeans} (cell G5) and $C_{T\text{-}shirts}$ (cell G6) for the maximum amount of money spent given the constraints?
8) How do these results compare with the same analysis done before in question 4 of this problem (see above)? Are these results similar?
9) What are the shadow prices for the different constraints? What is the maximum shadow price? If you change any constraint "1 unit", how much will the Objective function will change? Are there any biding constraints?

Exercise 4 (Adapted from Loucks and van Beek)

Consider a water-using industry that plans to obtain water from a groundwater aquifer. Two wellfield sites have been identified, *A* and *B*. The objective of this industry is to minimize the cost of pumping from the wells at sites *A* and *B*. The cost of pumping can be obtained by summing the cost of pumping at site *A* plus site *B* (*Cost A* + *Cost B*); both of them depend on the water extracted Q_A and Q_B from sites *A* and *B*, respectively. Cost A is expressed by the equation:

$$Cost\ A = 8 + \frac{40-8}{17} * Q_A$$

Cost B is expressed by the equation:

$$Cost\ B = 15 + \frac{26-15}{13} * Q_B$$

For the well at site *A*, the water extracted (Q_A) must be equal to or greater than 0. The maximum sustainable groundwater extraction at site *A* is equal to or less than 17 acre-feet/year.

For the well at site *B*, the water extracted (Q_B) must be equal to or greater than 0. The maximum sustainable groundwater extraction at site *B* is equal to or less than 13 acre-feet /year.
The water required to satisfy the industry is 22 acre-feet/year.

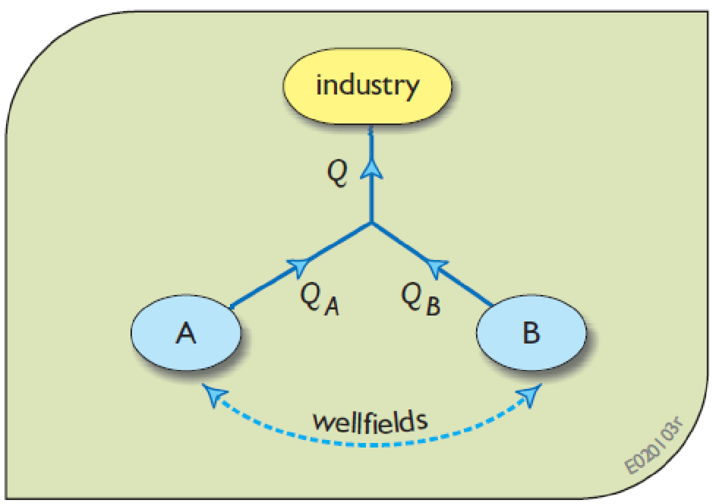


Figure 49

To be turned in:

Following the seven steps of linear programing (page 4 of this exercise)

1) Write down the *objective function* and the constraints
2) Draw the *feasible region*. You can use excel, or a simple piece of paper and scan it, or some engineering paper.
3) Obtain the vertices of the feasible region. Substitute the values of the vertices into the objective function. Submit a table showing the vertices of the feasible region as well as

the value of the objective function for each pair of Q_A and Q_B values (see slides 13 and 19 of the presentation of "Optimization").

4) Based on the previous analysis, what is the minimum cost of pumping for this industry?
5) What is the optimal water extraction Q_A and Q_B for the minimum cost of pumping?

Now, let's use the Open Solver model of excel to solve this problem. Use the file Ex_5.xlsx - tab "4 Minimize" to create a linear optimization model. You can use as initial values for QA (cell G5) and QB (cell G6) 10 and 10 AF/year , respectively. Calculate *Cost A* in cell D8 using the formula provided above. Similarly, calculate *Cost B* in cell D11 using the formula provided above. Don't forget to link the objective function in cell D5 by summing the *Cost A* (Cell D8) plus *Cost B* (Cell D11). When creating the linear optimization model in Open Solver, don't forget to select "Minimize" instead of "Maximize" (Figure 17).

6) Turn in a screenshot of your linear model, similar to Figure 44.
7) What is the minimum cost of pumping (the optimum value of the objective function) obtained once the model has been run (cell D5)?
8) What are the optimal water extractions Q_A (cell G5) and Q_B (cell G6) for the minimum cost of pumping?
9) How do these results compare with the same analysis done before in questions 4 and 5 of this problem (see above)? Are these results similar?